A Class of Preconditioners for Large Indefinite Linear Systems, as by-product of Krylov subspace Methods: Part II*

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(June 2011)

Abstract. In this paper we consider the parameter dependent class of preconditioners $M_h^\sharp(a,\delta,D)$ defined in the companion paper [6]. The latter was constructed by using information from a Krylov subspace method, adopted to solve the large symmetric linear system $Ax = b$. We first estimate the condition number of the preconditioned matrix $M_h^\sharp(a,\delta,D)A$. Then our preconditioners, which are independent of the choice of the Krylov subspace method adopted, proved to be effective also when solving sequences of slowly changing linear systems, in unconstrained optimization and linear algebra frameworks. A numerical experience is provided to give evidence of the performance of $M_h^\sharp(a,\delta,D)$.

Keywords: Preconditioners, large indefinite linear systems, large scale nonconvex optimization, Krylov subspace methods.

JEL Classification Numbers: C44, C61.

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* G.Fasano wishes to thank the Italian Ship Model Basin, CNR - INSEAN institute, for the indirect support.

1 Introduction

This paper is focused on both theoretical and computational results, for the parameter dependent class of preconditioners $M_h^\sharp(a, \delta, D)$, addressed in the companion paper [6]. The latter proposal is specifically suited for large scale problems, and our preconditioners are built using information collected by any Krylov subspace method, when solving the symmetric linear system $Ax = b$, $A \in \mathbb{R}^{n \times n}$ indefinite.

There is plenty of real applications and/or theoretical frameworks where the solution of large symmetric linear systems is amenable, including several contexts from nonlinear optimization. Examples of the latter contexts range from truncated Newton methods to KKT systems and interior point methods, not to mention the growing interest for PDE constrained optimization.

The class of preconditioners we propose is computationally cheap (in terms of the number of flops), and the construction of its members depends on the structural properties of matrix A. In particular, when A is positive definite, the Krylov subspace method adopted to solve the linear system provides, as by product, a factorization of a tridiagonal matrix, used to define our preconditioners. On the other hand, in case A is indefinite, the computation of the eigenpairs of a very small symmetric matrix (say at most 20×20) is performed, in order to construct the preconditioners. We remark that our parameter dependent preconditioners can be addressed by using a general Krylov subspace method. Moreover, we prove theoretical properties for the preconditioned matrix and we provide results which indicate how to possibly select the preconditioners parameters.

In this paper we experienced our preconditioners in the solution of linear systems from numerical analysis and in nonlinear optimization frameworks. In this regard, we preliminarily tested our proposal on significant linear systems from the literature, both including small/medium scale difficult linear systems and large systems. Then, we focused on *Newton–Krylov methods* (see [13] for a survey), and since our proposal may be extended to indefinite linear systems, we considered both convex and nonconvex problems.

The paper is organized as follows: in Section 2, we describe some properties of our class of preconditioners, recalling the results of the companion paper [6]. Section 3 is devoted to estimate the condition number of the preconditioned system matrix. In Section 4 we provide an extensive numerical experience using our preconditioners, and a section of conclusions and future work completes the paper.

As regards the notations, for a $n \times n$ real matrix M we denote with $\Lambda[M]$ the spectrum of M; I_k is the identity matrix of order k. We indicate with $\kappa(C)$ the condition number of the real matrix $C \in \mathbb{R}^{n \times n}$. Finally, with $C \succ 0$ we indicate that the matrix C is positive definite, $tr(C)$ and $det(C)$ are the trace and the determinant of C, while $\| \cdot \|$ denotes the Euclidean norm.

2 Our class of preconditioners

We recall here our class of preconditioners defined in the companion paper [6]. On this purpose, consider the *indefinite* linear system

$$Ax = b, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$ is *symmetric*, n is *large* and $b \in \mathbb{R}^n$. Suppose any Krylov subspace method is used for the solution of (2.1).

Assumption 2.1 *Let us consider any Krylov subspace method to solve the symmetric linear system (2.1). Suppose at step h of the Krylov method, with $h \le n-1$, the matrices $R_h \in \mathbb{R}^{n\times h}$, $T_h \in \mathbb{R}^{h\times h}$ and the vector $u_{h+1} \in \mathbb{R}^n$ are generated, such that*

$$AR_h = R_h T_h + \rho_{h+1} u_{h+1} e_h^T, \qquad \rho_{h+1} \in \mathbb{R}, \tag{2.2}$$

$$T_h = \begin{cases} V_h B_h V_h^T, & \text{if } T_h \text{ is indefinite} \\ \\ L_h D_h L_h^T, & \text{if } T_h \text{ is positive definite} \end{cases} \tag{2.3}$$

where

$R_h = (u_1 \cdots u_h), \;\; u_i^T u_j = 0, \;\; \|u_i\| = 1, \;\; 1 \le i \ne j \le h,$

$u_{h+1}^T u_i = 0, \;\; \|u_{h+1}\| = 1, \;\; 1 \le i \le h,$

T_h is symmetric and nonsingular, with eigenvalues $\mu_1, \ldots, \mu_h$ not all coincident

$B_h = diag_{1\le i\le h}\{\mu_i\}$, $V_h = (v_1 \cdots v_h) \in \mathbb{R}^{h\times h}$ orthogonal, (μ_i, v_i) is eigenpair of T_h,

$D_h \succ 0$ is diagonal, L_h is unit lower bidiagonal.

Then, using the notation (see also [8, 6])

$$|T_h| \stackrel{\text{def}}{=} \begin{cases} V_h |B_h| V_h^T, \;\; |B_h| = \; diag_{1\le i\le h}\{|\mu_i|\}, & \text{if } T_h \text{ is indefinite}, \\ \\ T_h, & \text{if } T_h \text{ is positive definite}, \end{cases}$$

the matrix $|T_h|$ is positive definite, for any choice of A and for any integer h. Now, recalling the matrix M_h, along with our class of preconditioners $M_h^\sharp(a, \delta, D)$

$$\begin{aligned} M_h^\sharp(a,\delta,D) &= D\left[I_n - (R_h \mid u_{h+1})\,(R_h \mid u_{h+1})^T\right] D^T \qquad\qquad h \le n-1, \\ &\quad + \;(R_h \mid Du_{h+1}) \left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array}\right)^{-1} (R_h \mid Du_{h+1})^T \qquad (2.4) \\ M_n^\sharp(a,\delta,D) &= R_n |T_n|^{-1} R_n^T, \qquad (2.5) \end{aligned}$$

both introduced in the companion paper [6], we have the following result.

Theorem 2.1 *Consider any Krylov-subspace method to solve the symmetric linear system (2.1), where A is indefinite. Suppose that Assumption 2.1 holds and the Krylov-subspace method performs $h \leq n$ iterations. Let $a \in \mathbb{R}$, $\delta \neq 0$, and let the matrix $D \in \mathbb{R}^{n \times n}$ be such that $[R_h \mid Du_{h+1} \mid DR_{n,h+1}]$ is nonsingular, where $R_{n,h+1}R_{n,h+1}^T = I_n - (R_h \mid u_{h+1})(R_h \mid u_{h+1})^T$. Then, we have the following properties:*

a) the matrix $M_h^\sharp(a,\delta,D)$ is symmetric. Furthermore,

- *when $h \leq n-1$, for any $a \in \mathbb{R} \setminus \{\pm\delta(e_h^T|T_h|^{-1}e_h)^{-1/2}\}$, $M_h^\sharp(a,\delta,D)$ is nonsingular. In addition, if $D = I_n$ then*

$$\det\left(M_h^\sharp(a,\delta,I_n)\right) = \delta^{-2h}\det(|T_h|^{-1})\left(1 - \frac{a^2}{\delta^2}e_h^T|T_h|^{-1}e_h\right)^{-1};$$

- *when $h = n$ the matrix $M_h^\sharp(a,\delta,D)$ is nonsingular. In addition, if $D = I_n$ then*

$$\det\left(M_n^\sharp(a,\delta,I_n)\right) = \det(|T_h|^{-1});$$

b) setting $D = I_n$ and $\delta = 1$ the matrix $M_h^\sharp(a,1,I_n)$ coincides with M_h^{-1};

c) for $|a| < |\delta|(e_h^T|T_h|^{-1}e_h)^{-1/2}$ the matrix $M_h^\sharp(a,\delta,D)$ is positive definite. Moreover, if $D = I_n$ the spectrum $\Lambda[M_h^\sharp(a,\delta,I_n)]$ is given by

$$\Lambda[M_h^\sharp(a,\delta,I_n)] = \Lambda\left[\left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array}\right)^{-1}\right] \cup \Lambda\left[I_{n-(h+1)}\right];$$

d) when $h \leq n-1$, $D = I_n$ and either $T_h \succ 0$ or T_h is indefinite

- *then $M_h^\sharp(a,\delta,I_n)A$ has at least $(h-3)$ singular values equal to $+1/\delta^2$;*
- *if $a = 0$ then the matrix $M_h^\sharp(0,\delta,I_n)A$ has at least $(h-2)$ singular values equal to $+1/\delta^2$;*

e) when $h = n$, then $M_n^\sharp(a,\delta,D) = M_n^{-1}$, $\Lambda[M_n] = \Lambda[|T_n|]$ and $\Lambda[M_n^{-1}A] = \Lambda[AM_n^{-1}] \subseteq \{-1,+1\}$, i.e. the n eigenvalues of the preconditioned matrix $M_h^\sharp(a,\delta,D)A$ are either $+1$ or -1.

Proof: See the companion paper [6]. □

3 On the condition number of matrix $M_h^\sharp(a,\delta,D)A$

In this section we want to estimate the condition number $\kappa(M_h^\sharp(a,\delta,D)A)$ of the unsymmetric matrix $M_h^\sharp(a,\delta,D)A$ (where $M_h^\sharp(a,\delta,D)$ is computed as in (2.4)-(2.5) and A is defined in (2.1)). We immediately have

$$\begin{aligned} \kappa(M_h^\sharp(a,\delta,D)A) &\stackrel{\text{def}}{=} \|M_h^\sharp(a,\delta,D)A\|_2 \cdot \|(M_h^\sharp(a,\delta,D)A)^{-1}\|_2 \\ &= \|M_h^\sharp(a,\delta,D)A\|_2 \cdot \|A^{-1}(M_h^\sharp(a,\delta,D))^{-1}\|_2, \end{aligned} \quad (3.1)$$

and we can prove the next technical lemma.

Lemma 3.1 *Let $C \in \mathbb{R}^{h \times h}$ be a symmetric and positive definite matrix. Let $0 < \omega_1 \leq \cdots \leq \omega_h$ be the ordered eigenvalues of C, with $\omega_1, \ldots, \omega_h$ not all coincident, and let $a \in \mathbb{R}$, $\delta \in \mathbb{R}$. Then, given the quantities*

$$\alpha = -\delta^2(h-1)\omega_1 + \delta^2 tr(C) + 1,$$

$$\beta = \frac{\delta^2 det(C)\left[1 - \frac{a^2}{\delta^2} e_h^T C^{-1} e_h\right]}{(\omega_h)^{h-1}},$$

we have

$$\alpha^2 - 4\beta > 0$$

In addition

$$\frac{[tr(C) - (h-1)\omega_1]\, \omega_h^{h-1}}{det(C)} > 1. \tag{3.2}$$

Proof: By the definition of α and β, and since $C \succ 0$, the condition $\alpha^2 - 4\beta \geq 0$ is satisfied if and only if

$$\delta^2 (e_h^T C^{-1} e_h)^{-1} \left[1 - \frac{\alpha^2 (\omega_h)^{h-1}}{4\delta^2 det(C)}\right] \leq a^2. \tag{3.3}$$

Now, observing that $\omega_1, \ldots, \omega_h$ are not all coincident, $\alpha > \delta^2 \omega_h + 1$ and for any $\omega_1 \geq 0$ we have $(\delta^2 \omega_1 + 1)^2 \geq 4\delta^2 \omega_1$, we obtain

$$\frac{\alpha^2 (\omega_h)^{h-1}}{4\delta^2 det(C)} \geq \frac{\alpha^2}{4\delta^2 \omega_1} > \frac{(\delta^2 \omega_h + 1)^2}{4\delta^2 \omega_1} \geq \frac{(\delta^2 \omega_1 + 1)^2}{4\delta^2 \omega_1} \geq 1, \tag{3.4}$$

so that (3.3) holds for any choice of a, which also implies that $\alpha^2 - 4\beta \geq 0$. Also observe that by (3.4) $\alpha^2 (\omega_h)^{h-1} / [4\delta^2 det(C)] > 1$, so that (3.3) can never be satisfied as an equality, i.e. $\alpha^2 - 4\beta \neq 0$ for any value of the parameter a.

Finally, note that since $det(C) = \omega_1 \cdot \cdots \cdot \omega_h$ we have

$$\omega_h^{h-1} > \frac{det(C)}{tr(C) - (h-1)\omega_1}, \tag{3.5}$$

inasmuch as $\omega_1, \ldots, \omega_h$ are not all coincident and

$$\frac{det(C)}{tr(C) - (h-1)\omega_1} \leq \frac{det(C)}{\omega_h} = \prod_{i=1}^{h-1} \omega_i < \omega_h^{h-1}.$$

As a consequence, we have the condition

$$\frac{[tr(C) - (h-1)\omega_1]\, \omega_h^{h-1}}{det(C)} > 1. \tag{3.6}$$

$\square$

In the following result we provide a general estimation of the condition number $\kappa(M_h^\sharp(a,\delta,D)A)$, which depends on the parameters 'δ' and 'a', and the matrix 'D' in (2.4). Note that for the sake of clarity, but with a little abuse of notation, in the sequel we directly indicate with $\mu_1,\dots,\mu_h$ the eigenvalues of $|T_h|$ and not the eigenvalues of T_h.

Proposition 3.2 *Consider the matrix $M_h^\sharp(a,\delta,D)$ in (2.4)-(2.5), with $h \le n-1$, where $|T_h|$ satisfies Assumption 2.1. Let $\mu_1 \le \cdots \le \mu_h$ be the (ordered) eigenvalues of $|T_h|$, where $\mu_1,\dots,\mu_h$ are not all coincident. Then, if*

$$|a| \; < |\delta|(e_h^T|T_h|^{-1}e_h)^{-1/2}, \qquad \delta \neq 0 \tag{3.7}$$

we have

$$\kappa(M_h^\sharp(a,\delta,D)A) \;\le\; \xi_h \cdot \kappa(N)^2 \cdot \kappa(A), \tag{3.8}$$

with

$$\xi_h = \frac{\max\left\{1, \frac{\gamma_h+(\gamma_h^2-4\sigma_h)^{1/2}}{2}\right\}}{\min\left\{1, \frac{\gamma_h-(\gamma_h^2-4\sigma_h)^{1/2}}{2}\right\}} \ge 1 \tag{3.9}$$

and

$$\gamma_h = -\delta^2(h-1)\mu_1 + \delta^2 tr(|T_h|) + 1$$

$$\sigma_h = \frac{\delta^2 det(|T_h|)\left[1 - \frac{a^2}{\delta^2}e_h^T|T_h|^{-1}e_h\right]}{(\mu_h)^{h-1}}.$$

In particular, when $D = I_n$ in (2.4) then $\kappa(M_h^\sharp A) \le \xi_h \cdot \kappa(A)$.

Proof: Let $\lambda_1 \le \cdots \le \lambda_{h+1}$ be the (ordered) eigenvalues of the matrix

$$\left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array}\right), \tag{3.10}$$

which is positive definite as long as condition (3.7) is fulfilled. Observe that by the identity

$$\left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array}\right) = \left(\begin{array}{c|c} I_h & 0 \\ \hline \frac{a}{\delta^2}e_h^T|T_h|^{-1} & 1 \end{array}\right) \left(\begin{array}{c|c} \delta^2|T_h| & 0 \\ \hline 0 & 1-\frac{a^2}{\delta^2}e_h^T|T_h|^{-1}e_h \end{array}\right) \left(\begin{array}{c|c} I_h & \frac{a}{\delta^2}|T_h|^{-1}e_h \\ \hline 0 & 1 \end{array}\right)$$

we have

$$det\left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array}\right) \;=\; \delta^{2h} det(|T_h|)\left[1 - \frac{a^2}{\delta^2}e_h^T|T_h|^{-1}e_h\right] \tag{3.11}$$

and $\delta^2|T_h|$ is the $h \times h$ upper left diagonal block of matrix (3.10). Therefore, by the Cauchy interlacing properties [4] between the sequences $\{\mu_j\}_{j=1,\dots,h}$ and $\{\lambda_i\}_{i=1,\dots,h+1}$ we have

$$\lambda_1 \le \delta^2\mu_1 \le \lambda_2 \le \delta^2\mu_2 \le \cdots \le \lambda_h \le \delta^2\mu_h \le \lambda_{h+1}. \tag{3.12}$$

By (3.10), (3.11) and (3.12) we can immediately infer the following intermediate results:

1. $\delta^2\mu_1 \le \lambda_i \le \delta^2\mu_h, \qquad\qquad i = 2,\dots,h$

2. $\sum_{i=1}^{h+1} \lambda_i = \delta^2 tr(|T_h|) + 1$

3. $\prod_{i=1}^{h+1} \lambda_i = \delta^{2h} det(|T_h|) \left[1 - \frac{a^2}{\delta^2} e_h^T |T_h|^{-1} e_h \right]$

From 1. we deduce that

$$\delta^2 (h-1)\mu_1 \;\leq\; \sum_{i=2}^{h} \lambda_i \;\leq \delta^2 (h-1)\mu_h,$$

so that from 2., 3., (3.12) and recalling that the matrix (3.10) is positive definite, we have

$$\max\left\{0, -\delta^2(h-1)\mu_h + \delta^2 tr(|T_h|) + 1\right\} \;\leq\; \lambda_1 + \lambda_{h+1} \;\leq\; -\delta^2(h-1)\mu_1 + \delta^2 tr(|T_h|) + 1$$

$$\frac{\delta^{2h} det(|T_h|) \left[1 - \frac{a^2}{\delta^2} e_h^T |T_h|^{-1} e_h \right]}{\delta^{2(h-1)} (\mu_h)^{h-1}} \;\leq\; \lambda_1 \cdot \lambda_{h+1} \;\leq\; \frac{\delta^{2h} det(|T_h|) \left[1 - \frac{a^2}{\delta^2} e_h^T |T_h|^{-1} e_h \right]}{\delta^{2(h-1)} (\mu_1)^{h-1}}. \tag{3.13}$$

From (3.13) (see also points (A) and (B) in Figure 3.1), in order to compute *bounds* λ_1

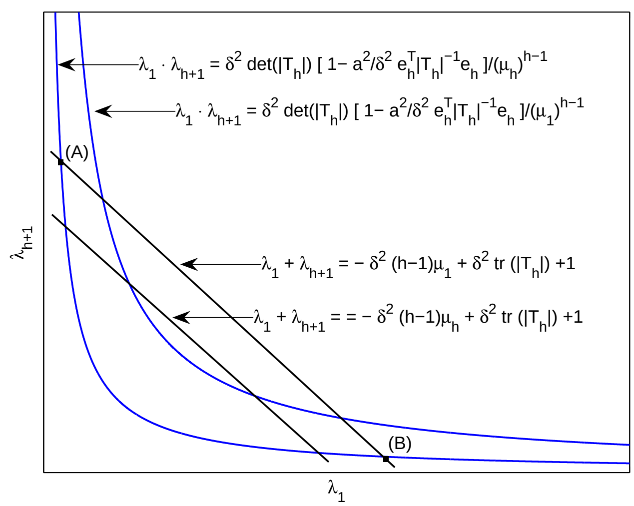

Figure 3.1: Relation between the eigenvalues λ_1 and λ_{h+1} of matrix (3.10).

$[\lambda_{h+1}]$ for the smallest [largest] eigenvalue of matrix (3.10), we have to solve the linear system (σ_h and γ_h are defined in the statement of this proposition)

$$\begin{cases} \tilde{\lambda}_1 + \tilde{\lambda}_{h+1} = \gamma_h \\ \tilde{\lambda}_1 \cdot \tilde{\lambda}_{h+1} = \sigma_h, \end{cases}$$

which yields

$$\tilde{\lambda}_1 = \frac{\gamma_h - (\gamma_h^2 - 4\sigma_h)^{1/2}}{2}$$
$$\tilde{\lambda}_{h+1} = \frac{\gamma_h + (\gamma_h^2 - 4\sigma_h)^{1/2}}{2}, \qquad (3.14)$$

provided that $\gamma_h^2 - 4\sigma_h \geq 0$. However, the latter condition directly holds from Lemma 3.1. Now, observe that from Theorem 2.1, setting $N = [R_h \mid Du_{h+1} \mid DR_{n,h+1}]$ (where N is nonsingular by hypothesis), for $h \leq n-1$ the preconditioners $M_h^\sharp(a,\delta,D)$ may be rewritten as

$$M_h^\sharp(a,\delta,D) = N \left[\begin{array}{c|c} \left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array} \right)^{-1} & 0 \\ \hline 0 & I_{n-(h+1)} \end{array} \right] N^T, \qquad h \leq n-1. \qquad (3.15)$$

As a consequence, setting

$$W_h = \left[\begin{array}{c|c} \left(\begin{array}{c|c} \delta^2|T_h| & ae_h \\ \hline ae_h^T & 1 \end{array} \right) & 0 \\ \hline 0 & I_{n-(h+1)} \end{array} \right],$$

we have for the smallest [largest] eigenvalue λ_m [λ_M] of matrices W_h and W_h^{-1} the expressions

$$\begin{cases} \lambda_m(W_h) = \min\{1, \lambda_1\} \\ \\ \lambda_M(W_h) = \max\{1, \lambda_{h+1}\} \end{cases}$$
$$\begin{cases} \lambda_m(W_h^{-1}) = \frac{1}{\max\{1,\lambda_{h+1}\}} \\ \\ \lambda_M(W_h^{-1}) = \frac{1}{\min\{1,\lambda_1\}}. \end{cases}$$

Thus, if $\lambda_m(A)$ [$\lambda_m(A^{-1})$] and $\lambda_M(A)$ [$\lambda_M(A^{-1})$] are the smallest [largest] eigenvalue of matrix A [A^{-1}] respectively, from (3.15) we have

$$\|M_h^\sharp(a,\delta,D)A\| \;\leq\; \lambda_M(A)\cdot\|N\|^2\cdot\lambda_M(W_h^{-1}) \;\leq\; \lambda_M(A)\cdot\|N\|^2\cdot\frac{1}{\min\{1,\lambda_1\}}$$

and

$$\begin{aligned} \|(M_h^\sharp(a,\delta,D)A)^{-1}\| &= \|A^{-1}(M_h^\sharp(a,\delta,D))^{-1}\| \;\leq\; \lambda_M(A^{-1})\cdot\|N^{-1}\|^2\cdot\lambda_M(W_h) \\ &\leq \frac{1}{\lambda_m(A)}\cdot\|N^{-1}\|^2\cdot\max\{1,\lambda_{h+1}\}, \end{aligned}$$

so that from (3.14)

$$\kappa\left(M_h^\sharp(a,\delta,D)A\right) = \|M_h^\sharp(a,\delta,D)A\|\cdot\|(M_h^\sharp(a,\delta,D)A)^{-1}\| \;\leq\; \frac{\max\left\{1,\tilde{\lambda}_{h+1}\right\}}{\min\left\{1,\tilde{\lambda}_1\right\}}\kappa(N)^2\kappa(A),$$

which is relation (3.8). Finally, when $D = I_n$ in (2.4) then $\kappa(N) = 1$. □

In order to better specify the bound (3.8) we can now prove the next lemma.

Lemma 3.3 *Let us consider the hypotheses of Proposition 3.2 and the quantity ξ_h defined in (3.9). Then, for any choice of 'δ' and 'a' satisfying (3.7) we have*

$$\xi_h = \frac{\gamma_h + (\gamma_h^2 - 4\sigma_h)^{1/2}}{\gamma_h - (\gamma_h^2 - 4\sigma_h)^{1/2}}. \tag{3.16}$$

Proof: The proof consists to analyze the following three cases:

1. $\gamma_h < 2$ (i.e. $\delta^2 < 1/[tr(|T_h|) - (h-1)\mu_1]$)
2. $\gamma_h = 2$ (i.e. $\delta^2 = 1/[tr(|T_h|) - (h-1)\mu_1]$)
3. $\gamma_h > 2$ (i.e. $\delta^2 > 1/[tr(|T_h|) - (h-1)\mu_1]$)

In case 1. is satisfied, observe that the inequality

$$\frac{\gamma_h + (\gamma_h^2 - 4\sigma_h)^{1/2}}{2} < 1$$

cannot hold, since (consider that $\gamma_h - 2 < 0$ and see Lemma 3.1) it requires that

$$\gamma_h < 1 + \sigma_h \qquad \text{iff} \qquad a^2 < \left[1 - \frac{(\gamma_h - 1)\mu_h^{h-1}}{\delta^2 det(|T_h|)}\right] \frac{\delta^2}{e_h^T |T_h|^{-1} e_h}$$

which can hold only if

$$\frac{(\gamma_h - 1)\mu_h^{h-1}}{\delta^2 det(|T_h|)} \leq 1$$

or equivalently

$$\delta^2 \geq \frac{(\gamma_h - 1)\mu_h^{h-1}}{det(|T_h|)}.$$

However, the last inequality cannot hold because it is equivalent to

$$1 \geq \frac{[tr(|T_h|) - (h-1)\mu_1]\mu_h^{h-1}}{det(|T_h|)},$$

which cannot be satisfied from Lemma 3.1. Moreover, in case 1., also

$$\frac{\gamma_h - (\gamma_h^2 - 4\sigma_h)^{1/2}}{2} > 1$$

cannot hold, since $\gamma_h - 2 < 0$. Therefore, when $\gamma_h < 2$ relation (3.16) holds.

The case 2. is pretty similar to the case 1., so that again (3.16) follows almost immediately.

In case 3., the inequality

$$\frac{\gamma_h + (\gamma_h^2 - 4\sigma_h)^{1/2}}{2} < 1$$

cannot hold since it is equivalent to $(\gamma_h^2 - 4\sigma_h)^{1/2} < 2 - \gamma_h < 0$. Moreover, from Lemma 3.1 and considering that $\gamma_h - 2 > 0$, the condition

$$\frac{\gamma_h - (\gamma_h^2 - 4\sigma_h)^{1/2}}{2} > 1$$

can be satisfied if

$$\gamma_h < 1 + \sigma_h \quad \text{iff} \quad a^2 < \left[1 - \frac{(\gamma_h - 1)\mu_h^{h-1}}{\delta^2 det(|T_h|)}\right] \frac{\delta^2}{e_h^T |T_h|^{-1} e_h},$$

which holds only if

$$\frac{(\gamma_h - 1)\mu_h^{h-1}}{\delta^2 det(|T_h|)} \leq 1$$

or equivalently

$$\delta^2 \geq \frac{(\gamma_h - 1)\mu_h^{h-1}}{det(|T_h|)}.$$

However, since $\gamma_h - 1 = tr(|T_h|) - (h-1)\mu_1$, the last inequality is again equivalent to

$$1 \geq \frac{[tr(|T_h|) - (h-1)\mu_1]\mu_h^{h-1}}{det(|T_h|)}$$

which cannot hold from Lemma 3.1. Thus relation (3.16) holds.

□

Lemma 3.4 *Consider the matrix* $M_h^\sharp(a, \delta, D)$ *in (2.4)-(2.5), with* $h \leq n$. *Let* $\mu_1 \leq \cdots \leq \mu_h$ *be the (ordered) eigenvalues of* $|T_h|$, *with* $\mu_1, \cdots, \mu_h$ *not all coincident, and let the parameters 'a' and 'δ' satisfy condition (3.7). Then, for any choice of the matrix D in (2.4)*

- *the coefficient* ξ_h *in (3.9) increases when* $|a| \to \rho$, *with* $\rho = |\delta|(e_h^T |T_h|^{-1} e_h)^{-1/2}$, *and*

$$\lim_{|a| \to \rho} \xi_h = +\infty$$

- *the coefficient* ξ_h *in (3.9) attains its minimum when* $a = 0$, *and for* $a = 0$ *we have for the coefficient* ξ_h *in (3.9) the expression*

$$\xi_h = \frac{\gamma_h + \left(\gamma_h^2 - 4\frac{\delta^2 det(|T_h|)}{(\mu_h)^{h-1}}\right)^{1/2}}{\gamma_h - \left(\gamma_h^2 - 4\frac{\delta^2 det(|T_h|)}{(\mu_h)^{h-1}}\right)^{1/2}}. \tag{3.17}$$

Proof: Observe that when $|a| \to \rho$ then in the expression (3.9) of ξ_h we have $\sigma_h \to 0$, along with $\gamma_h - (\gamma_h^2 - 4\sigma_h)^{1/2} \to 0$ and $\gamma_h + (\gamma_h^2 - 4\sigma_h)^{1/2} \to 2\gamma_h$, with $\gamma_h > 1$. Thus, since from Lemma 3.1 $\gamma_h - 4\sigma_h \geq 0$, Lemma 3.3 ensures that ξ_h satisfies (3.16), so that

$\bar{\xi}_h$ increases as $|a| \to \rho$, with $\lim_{|a|\to\rho} \bar{\xi}_h = +\infty$. Moreover, from (3.16) and since ξ_h is a continuous function of the parameter 'a' (see (3.7)), we have

$$\frac{\partial \xi_h}{\partial a} = \frac{\partial \xi_h}{\partial \sigma_h} \cdot \frac{\partial \sigma_h}{\partial a} = \frac{-2\gamma_h}{[\gamma_h - (\gamma_h^2 + 4\sigma_h)^{1/2}]^2 (\gamma_h^2 - 4\sigma_h)^{1/2}} \cdot \frac{-2a \cdot det(|T_h|) e_h^T |T_h|^{-1} e_h}{(\mu_h)^{h-1}},$$

so that for $|a| < \rho$ we have $sgn\{\partial \xi_h / \partial a\} = sgn\{a\}$, which implies that ξ_h attains its minimum for $a = 0$.

Finally, by Lemma 3.1 $\gamma_h^2 - 4\sigma_h \geq 0$ for any choice of a satisfying (3.7), and when $a = 0$ it is $\sigma_h = \delta^2 det(|T_h|)/(\mu_h)^{h-1}$. Thus, from Lemma 3.3 the value of ξ_h when $a = 0$ is given by

$$\xi_h = \frac{\gamma_h + \left(\gamma_h^2 - 4\frac{\delta^2 det(|T_h|)}{(\mu_h)^{h-1}}\right)^{1/2}}{\gamma_h - \left(\gamma_h^2 - 4\frac{\delta^2 det(|T_h|)}{(\mu_h)^{h-1}}\right)^{1/2}},$$

so that (3.17) holds. □

Remark 3.1 By (3.17) we observe that as expected, the parameter 'δ' both affects the distribution of the singular values of $M_h^\sharp(a,\delta,D)A$ (see item *d*) of Theorem 2.1), and also its condition number $\kappa(M_h^\sharp(a,\delta,D)A)$, when computed according with (3.1).

4 Preliminary numerical results

In order to preliminarily test our proposal on a general framework, where no information is known about the sparsity pattern of the matrix A, we used our parameter dependent class of preconditioners $M_h^\sharp(a,\delta,D)$, setting $\delta = 1$ and $D = I_n$.

In our numerical experience we obtain even better results w.r.t. the theory. Indeed, all the results assessed in Theorem 2.1 for the *singular values* of the (possibly) unsymmetric matrix $M_h^\sharp(a,\delta,D)A$, seem to hold in practice also for the *eigenvalues* of $M_h^\sharp(a,\delta,D)A$ (we recall that since $M_h^\sharp(a,\delta,D) \succ 0$ then $\Lambda[M_h^\sharp(a,\delta,D)A] \equiv \Lambda[M_h^\sharp(a,\delta,D)^{1/2} A M_h^\sharp(a,\delta,D)^{1/2}]$), so that $M_h^\sharp(a,\delta,D)A$ has only real eigenvalues. In order to test the class of preconditioners (2.4)-(2.5), we used 4 different sets of test problems.

First, we considered a set of symmetric linear systems as in (2.1), where the number of unknowns n is set as $n = 1000$, and the matrix A has also a moderate condition number. We simply wanted to experience how our class of preconditioners modifies the condition number of A. In particular (see also [7]), a possible choice for the latter class of matrices is given by

$$A = \{a_{i,j}\}, \qquad a_{ij} \in U[-10, 10], \quad i, j = 1, \ldots, n, \tag{4.1}$$

where $a_{i,j} = a_{j,i}$ are random entries in the uniform distribution $U[-10, 10]$, between -10 and $+10$. Then, also the vector b in (2.1) is computed randomly with entries in the set $U[-10, 10]$. We computed the preconditioners (2.4)-(2.5) by using the *Conjugate Gradient*

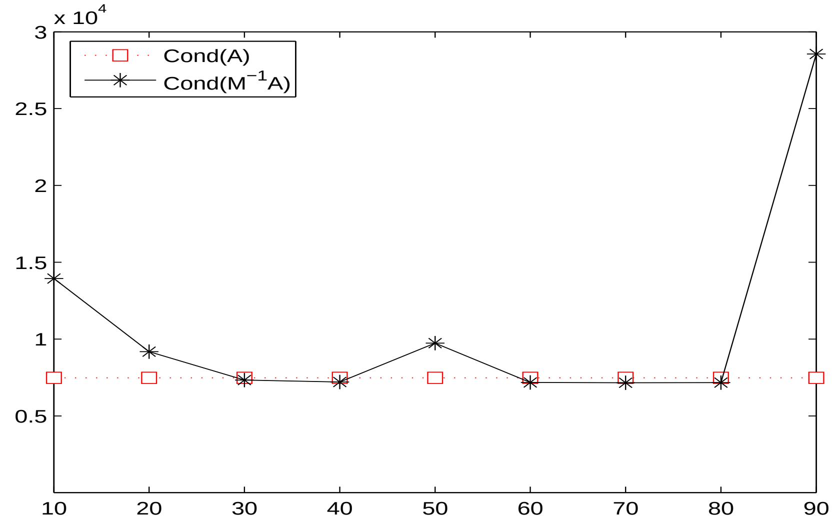

Figure 4.1: The condition number of matrix A ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)A$ ($Cond(M^{-1}A)$), when $h \in \{10, 20, 30, 40, 50, 60, 70, 80, 90\}$, and A is randomly chosen with entries in the uniform distribution $U[-10, 10]$.

(CG) method [16], which is one of the most popular Krylov subspace methods to solve (2.1) [9]. We remark that the CG is often used also in case the matrix A is indefinite, though it can prematurely stop. As an alternative choice, in order to satisfy Assumption 2.1 with A indefinite, we can use the Lanczos process [11], MINRES methods [15] or Planar-CG methods [5]. In (2.4) we set the parameter h in the range

$$h \in \{\ 20\ ,\ 30\ ,\ 40\ ,\ 50\ ,\ 60\ ,\ 70\ ,\ 80\ ,\ 90\ \},$$

and we preliminarily chose $a = 0$ (though other choices of the parameter 'a' yield similar results), which satisfied items a) and c) of Theorem 2.1. We plotted in Figure 4.1 the condition number $\kappa(A)$ of A ($Cond(A)$), along with the condition number $\kappa(M_h^\sharp(0,1,I)A)$ of $M_h^\sharp(0,1,I)A$ ($Cond(M^{-1}A)$): in both cases the condition number κ is calculated by preliminarily computing the eigenvalues $\lambda_1, \ldots, \lambda_n$ (using `Matlab` [1] routine `eigs()`) of A and $M_h^\sharp(0,1,I)A$ respectively, then obtaining the ratio

$$\kappa = \frac{\max_i\ |\lambda_i|}{\min_i\ |\lambda_i|}.$$

Evidently, numerical results confirm that the *order* of the condition number of A is pretty similar to that of the condition number of $M_h^\sharp(0,1,I)A$. This indicates that if the preconditioners (2.4) are used as a tool to solve (2.1), then most preconditioned iterative methods which are sensible to the condition number (e.g. the Krylov subspace methods), on average are not expected to perform worse with respect to the unpreconditioned case. However, it is important to remark that the spectrum $\Lambda[M_h^\sharp(0,1,I)A]$ tends to be shifted with respect

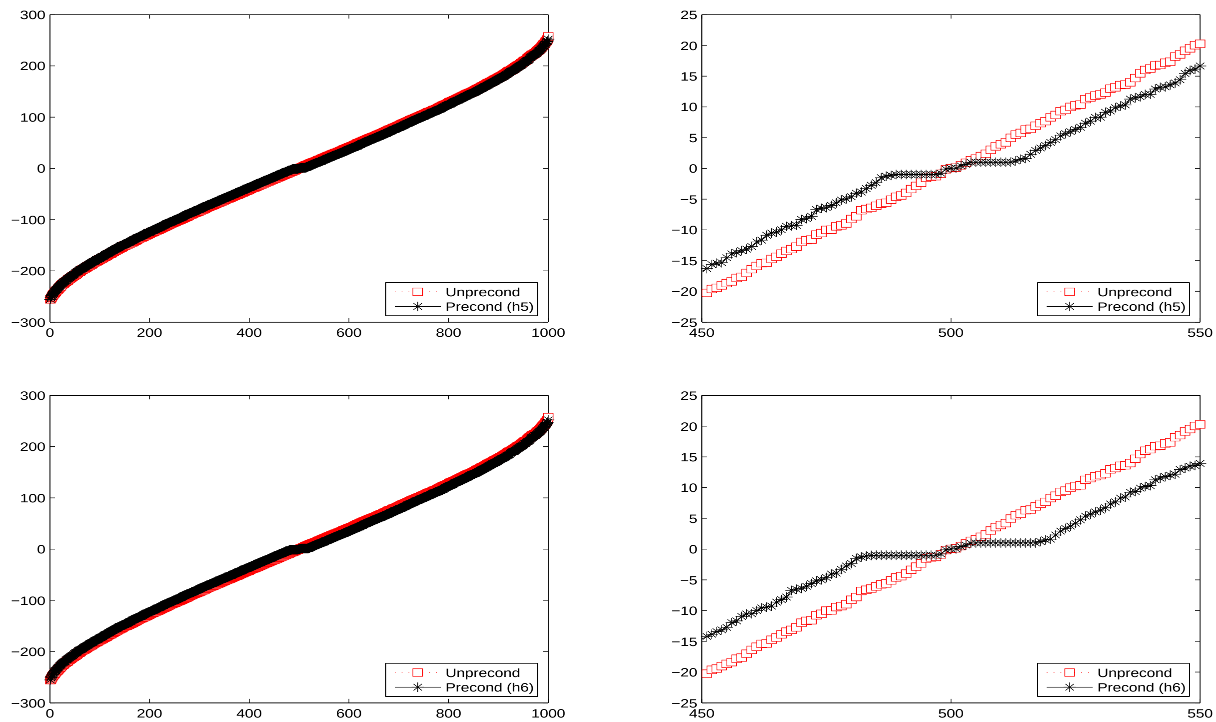

Figure 4.2: Comparison between the *full/detailed* spectra (left/right figures) $\Lambda[A]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A]$ (*Precond*), with A randomly chosen (eigenvalues are sorted for simplicity); without loss of generality we show the results for the values $h = h5 = 20$ and $h = h6 = 30$. The intermediate eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A]$, whose absolute value is larger than 1, are in general smaller than the corresponding eigenvalues in $\Lambda[A]$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A]$ are more clustered near $+1$ or -1 than those in $\Lambda[A]$.

to $\Lambda[A]$, inasmuch as the eigenvalues in $\Lambda[A]$ whose absolute value is larger than $+1$ tend to be scaled in $\Lambda[M_h^\sharp(0,1,I)A]$ (see Figure 4.2). The latter property is an appealing result, since the eigenvalues of $M_h^\sharp(0,1,I)A$ will be 'more clustered'. The latter phenomenon has been better investigated by introducing other sets of test problems, described hereafter.

In a second experiment we generated the set of matrices A such that

$$A = H\mathcal{D}H, \tag{4.2}$$

where $H \in \mathbb{R}^{n\times n}$, $n = 500$, is an Householder transformation given by $H = I - 2vv^T$, with $v \in \mathbb{R}^n$ a unit vector, randomly chosen. The matrix $\mathcal{D} \in \mathbb{R}^{n\times n}$ is diagonal (so that its non-zero entries are also eigenvalues of A, while each column of H is also an eigenvector of A). The matrix $\mathcal{D}$ is such that its $\texttt{perc} \cdot n$ eigenvalues are larger (about one order of magnitude) than the remaining $(1 - \texttt{perc}) \cdot n$ eigenvalues (we set without loss of generality perc $= 0.3$). Finally, again we computed the preconditioners (2.4)-(2.5) by using the CG, setting the starting point x_0 so that the initial residual $b - Ax_0$ was a linear combination (with coefficients -1 and $+1$ randomly chosen) of *all* the n eigenvectors of A. We strongly highlight that the latter choice of x_0 is expected to be not favorable when applying the CG, to build our preconditioners. In the latter case the CG method is indeed expected to perform exactly n iterations before stopping (see also [14, 16]), so that the matrices (4.2) may be significant to test the effectiveness of our preconditioners, in case of *small values* of h (broadly speaking, h small implies that the preconditioner contains correspondingly a little information on the inverse matrix A^{-1}). We compared the spectra $\Lambda[A]$ and $\Lambda[M_h^\sharp(a,1,I)A]$, in order to verify again how the preconditioners (2.4) are able to *cluster the eigenvalues of* A. Following exactly the choice in [12], in order to test our proposal also on a different range of values for the parameter h, we set

$$h \in \{ \ 4 \ , \ 8 \ , \ 12 \ , \ 16 \ , \ 20 \ \}.$$

The results are given in Figure 4.3 (*full comparisons*) which includes all the 500 eigenvalues, and Figure 4.4 (*details*) which includes only the eigenvalues from the 410-th to the 450-th. Observe that our preconditioners are able to shift the largest absolute eigenvalues of A towards -1 or $+1$, so that the clustering of the eigenvalues is enhanced when the parameter h increases. For any value of h the matrix A is (randomly) recomputed from scratch, according with relation (4.2). This explains while in the five plots of Figures 4.3-4.4 the spectrum of A changes. Again, a behavior very similar to Figures 4.3-4.4 is obtained also using different values for the parameter 'a'.

We used another small set of test problems, obtained by considering a couple of linear systems as (2.1), described in [12, 3] and therein references, which come up from finite element problems. We addressed the latter linear systems as $A_0x = b_0$ (*from one-dimensional model, consisting of a line of two-node elements with support conditions at both ends, and a linearly varying body force*) and $A_1x = b_1$ (where A_1 is the *stiffness matrix from a two-dimensional finite element model of a cantilever beam*) respectively [12]. The spectral properties of both the matrices A_0 and A_1 are extensively described in [12]. In particular $A_0 \in \mathbb{R}^{50\times 50}$ is positive definite with condition number $\kappa(A_0) = 0.20E + 10$ and with a suitable pattern of clustering of the eigenvalues; similarly, $A_1 \in \mathbb{R}^{170\times 170}$ is also positive definite, with condition number $\kappa(A_1) = 0.13E + 9$ and a different pattern of eigenvalues clustering. In addition, we have

$$b_0 = \begin{pmatrix} 0 \\ 200/49 \\ 300/49 \\ \vdots \\ 4900/49 \\ 0 \end{pmatrix},$$

$$b_1 = 0, \quad \text{but} \quad b_1(34) = b_1(68) = b_1(102) = b1(136) = b1(170) = -8000,$$

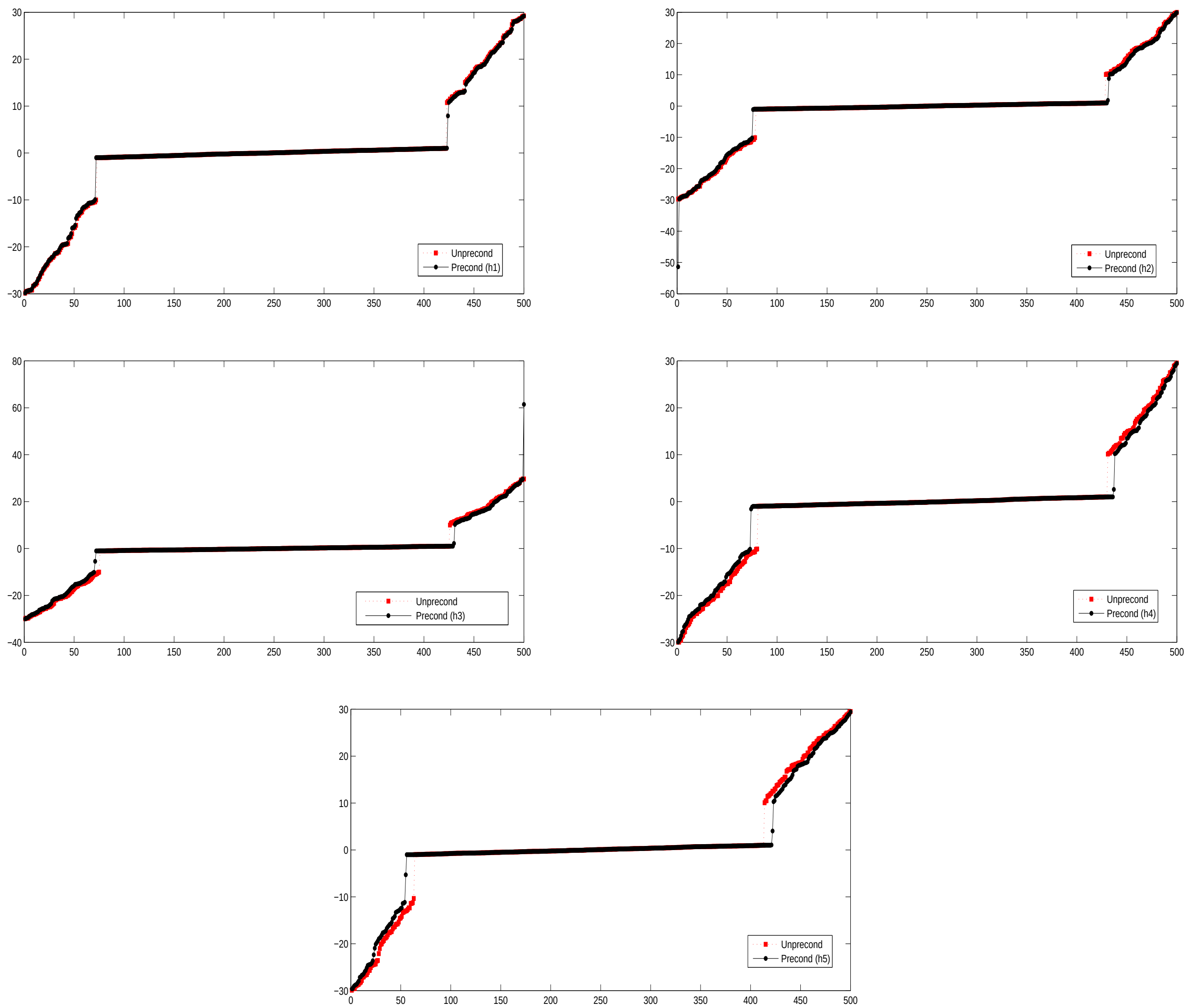

Figure 4.3: Comparison between the *full spectra* $\Lambda[A]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A]$ (*Precond*), with A nonsingular and given by (4.2) (eigenvalues are sorted for simplicity); we used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$), setting $n = 500$. The large eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A]$ are in general smaller (in modulus) than the corresponding large eigenvalues in $\Lambda[A]$. A 'flatter' piecewise-line of the eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A]$ indicates that the eigenvalues tend to cluster around -1 and $+1$, according with the theory.

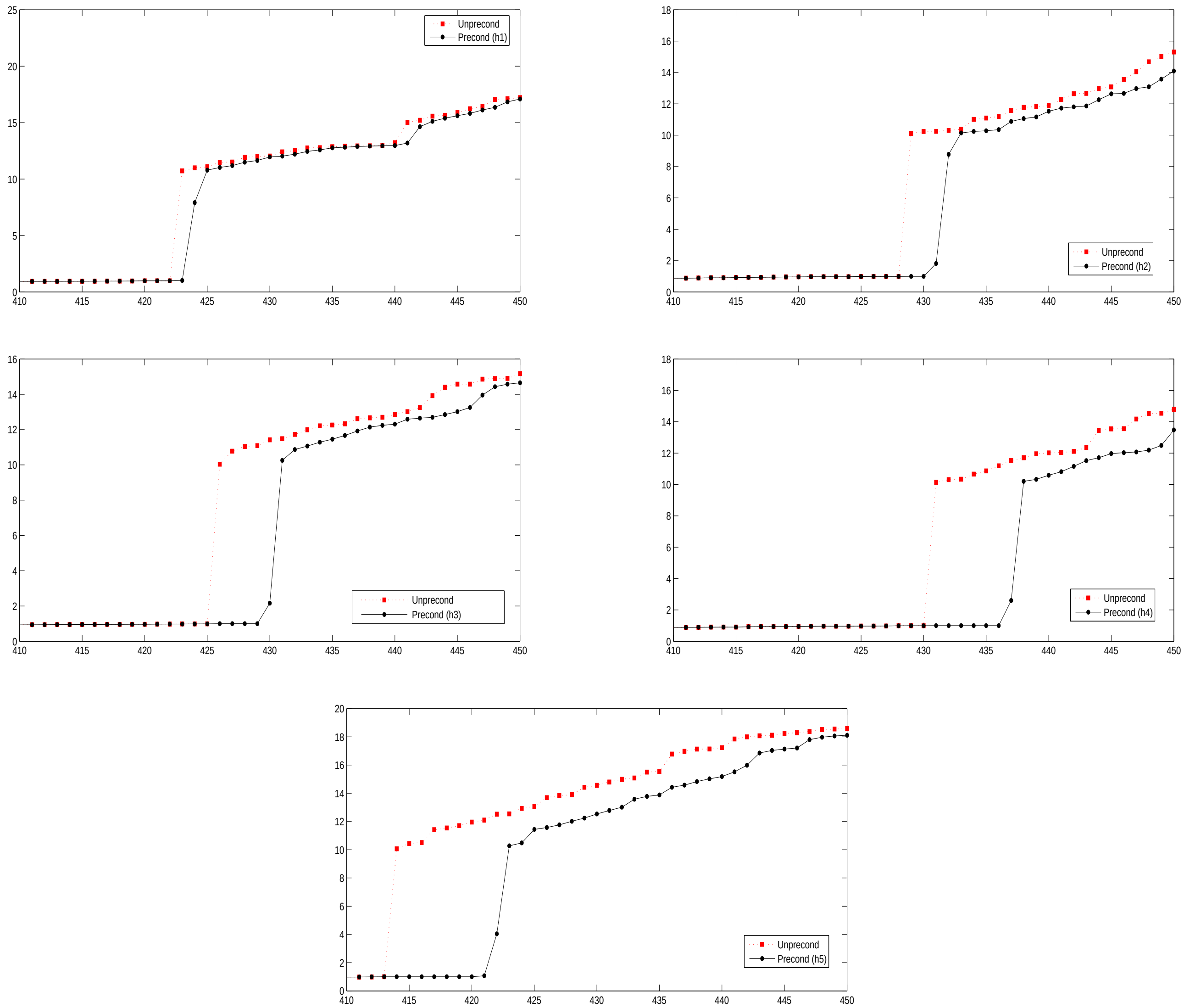

Figure 4.4: Comparison between a *detail of the spectra* $\Lambda[A]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A]$ (*Precond*), with A nonsingular and given by (4.2) (eigenvalues are sorted for simplicity; we used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$), setting $n = 500$. The large eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A]$ are in general smaller (in modulus) than the corresponding large eigenvalues in $\Lambda[A]$. A 'flatter' piecewise-line of the eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A]$ indicates that the eigenvalues tend to cluster around -1 and $+1$, according with the theory.

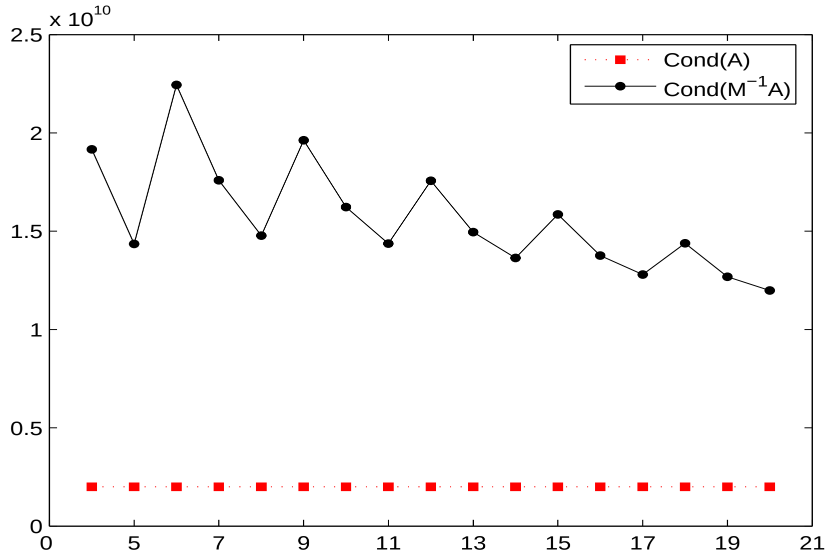

Figure 4.5: The condition number of matrix A_0 ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)A_0$ ($Cond(M^{-1}A)$), when $4 \leq h \leq 20$. The condition number of A_0 is slightly larger than the condition number of $M_h^\sharp(0,1,I)A_0$, for any value of the parameter h. The starting point of the CG is $x_0 = 0$.

and the CG is again used to compute the preconditioner $M_h^\sharp(0,1,I)$, adopting both the starting points $x_0 = 0$ and $x_0 = 100$, $e = (1 \cdots 1)^T$, as indicated in [12].
We have computed our class of preconditioners for the linear systems $A_0x = b_0$ and $A_1x = b_1$, with $a = 1$ and $h \in \{4, 8, 12, 16, 20\}$. The effect of the preconditioner on the condition number of matrix A_0 is plotted in Figure 4.5 ($Cond(A)$ / $Cond(M^{-1}A)$ with $x_0 = 0$) and Figure 4.6 ($Cond(A)$ / $Cond(M^{-1}A)$ with $x_0 = 100e$). Furthermore, the comparison between the spectra $\Lambda[A_0]$ and $\Lambda[M_h^\sharp(0,1,I)A_0]$, for different values of h, is given in Figure 4.7 ($x_0 = 0$) and Figure 4.8 ($x_0 = 100e$). Similarly, the comparison between the *preconditioned/unpreconditioned* matrix A_1 using the preconditioner $M_h^\sharp(0,1,I)$, with $h \in \{4, 8, 12, 16, 20\}$ and $a = 1$, is plotted in Figures 4.9 - 4.12. Here, though the preconditioner can slightly deteriorate the condition number $\kappa(A_1)$ (the case $x_0 = 0$), the effect of *clustering the eigenvalues* is still evident, since the intermediate eigenvalues are uniformly scaled.

To complete our numerical experience we tested our class of preconditioners in an optimization framework. In particular, we considered an unconstrained optimization problem, which was solved using the linesearch-based truncated Newton method in Table 4.1, where the solution of the symmetric linear system (Newton's equation) $\nabla^2 f(x_k)d = -\nabla f(x_k)$ is required. We considered several smooth optimization problems from `CUTEr` [10] collection, and for each problem we applied the truncated Newton method in Table 4.1. At the outer

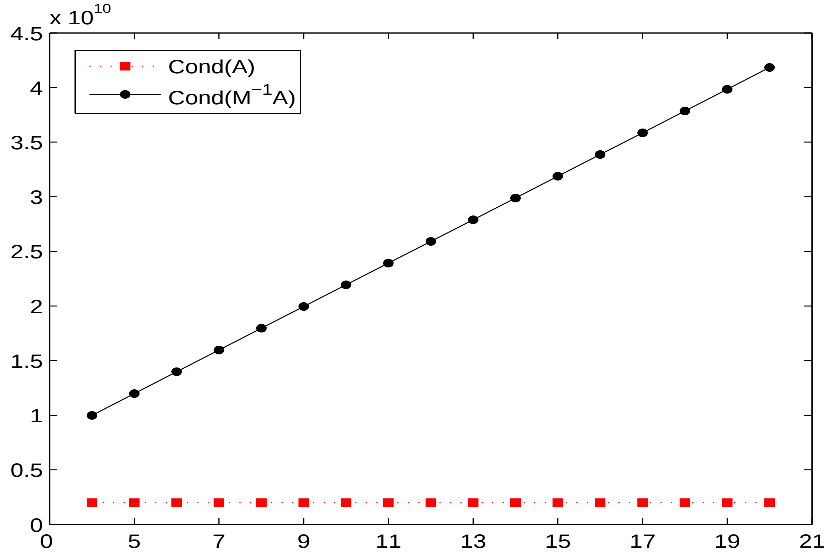

Figure 4.6: The condition number of matrix A_0 ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)A_0$ ($Cond(M^{-1}A)$), when $4 \le h \le 20$. The condition number of A_0 is slightly larger than the condition number of $M_h^\sharp(0,1,I)A_0$, for any value of the parameter h. The starting point of the CG is $x_0 = 100e$.

Table 4.1: The linesearch-based truncated Newton method we adopted.

Set $x_0 \in \mathbb{R}^n$
Set $\eta_k \in [0,1)$ for any k, with $\{\eta_k\} \to 0$
OUTER ITERATIONS
for $k = 0, 1, \ldots$
    Compute $\nabla f(x_k)$; if $\|\nabla f(x_k)\|$ is small then STOP
        INNER ITERATIONS
        Compute d_k which approximately solves $\nabla^2 f(x_k) d = -\nabla f(x_k)$
        and satisfies the *truncation rule*
$$\|\nabla^2 f(x_k) d_k + \nabla f(x_k)\| \le \eta_k \|\nabla f(x_k)\|$$
    Compute the steplength α_k by an Armijo-type linesearch scheme
    Update $x_{k+1} = x_k + \alpha_k d_k$
endfor

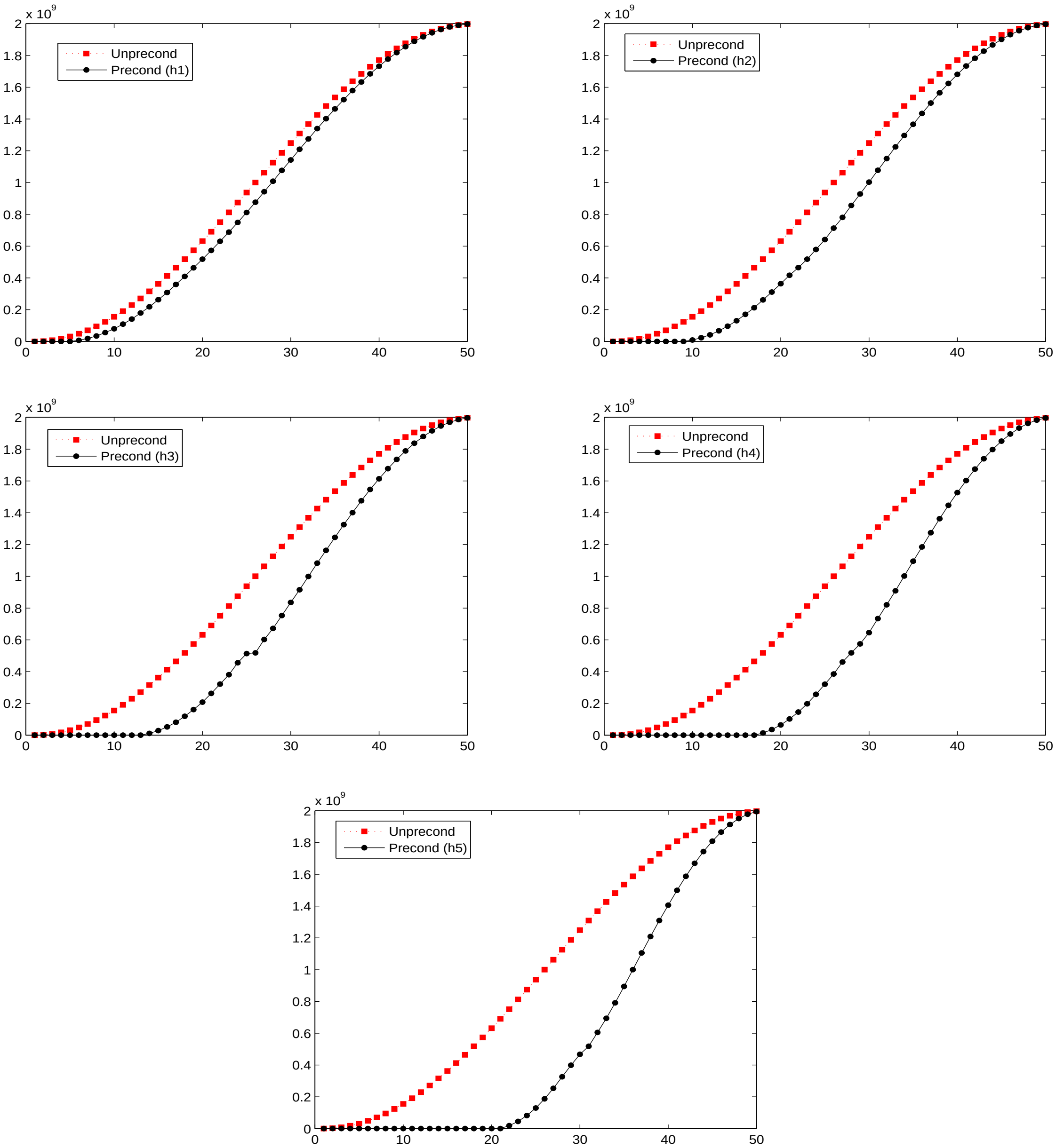

Figure 4.7: Comparison between the *full spectra* $\Lambda[A_0]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A_0]$ (*Precond*), with A_0 nonsingular (eigenvalues are sorted for simplicity); we used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$). The eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A_0]$ are in general smaller than the corresponding eigenvalues in $\Lambda[A_0]$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A_0]$ are also more clustered near $+1$. The starting point of the CG is $x_0 = 0$.

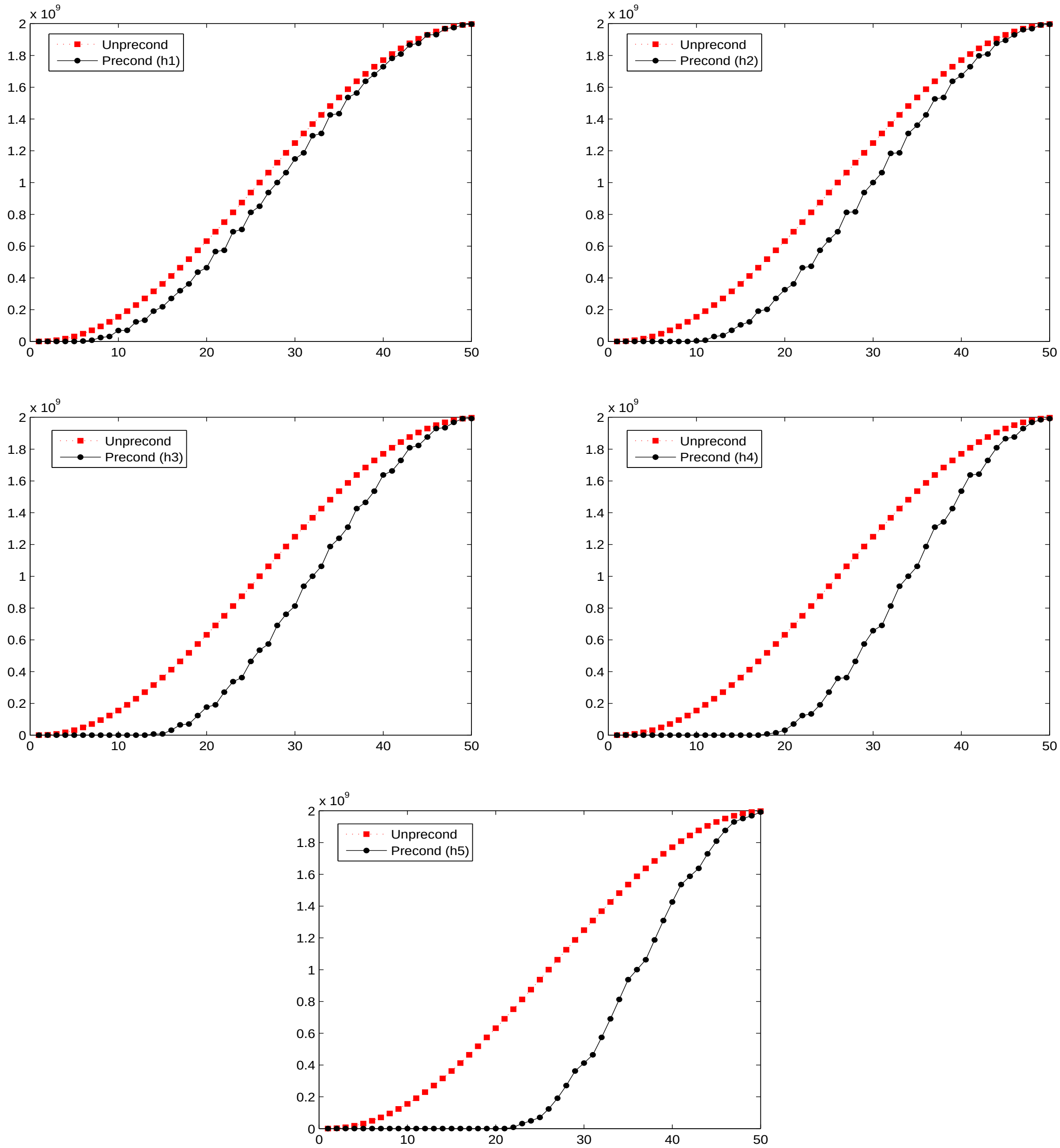

Figure 4.8: Comparison between the *full spectra* $\Lambda(A_0)$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A_0]$ (*Precond*), with A_0 nonsingular (eigenvalues are sorted for simplicity); we used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$). The eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A_0]$ are in general smaller than the corresponding eigenvalues in $\Lambda[A_0]$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A_0]$ are also more clustered near $+1$. The starting point of the CG is $x_0 = 100e$.

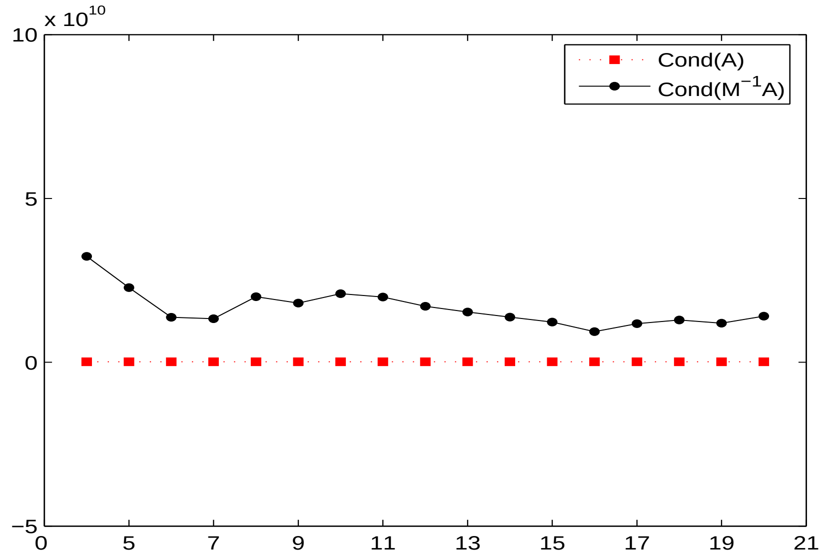

Figure 4.9: The condition number of matrix A_1 ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)A_1$ ($Cond(M^{-1}A)$), when $4 \le h \le 20$. The condition number of A_1 is now slightly *smaller* than the condition number of $M_h^\sharp(0,1,I)A_1$, for any value of the parameter h. The starting point of the CG is $x_0 = 0$.

iteration k we computed the preconditioner $M_h^\sharp(a,1,I)$, with $h \in \{4,8,12,16,20\}$, by using the CG to solve the equation $\nabla^2 f(x_k)d = -\nabla f(x_k)$. Then, we adopted $M_h^\sharp(0,1,I)$ as a preconditioner for the solution of Newton's equation of the subsequent iteration

$$\nabla^2 f(x_{k+1})d = -\nabla f(x_{k+1}).$$

The iteration index k was randomly chosen, in such a way that $\|x_{k+1} - x_k\|$ was *small* (i.e. the entries of the Hessian matrices $\nabla^2 f(x_k)$ and $\nabla^2 f(x_{k+1})$ are not expected to differ significantly). For simplicity we just report the results on two test problems, using $n = 1000$, in the set of all the optimization problems experienced. Very similar results were obtained for almost all the test problems. In Figures 4.13-4.14 we consider the problem `NONCVXUN`; without loss of generality we only show the numerical results for $h = 16$. Observe that since x_{k+1} is close to x_k (i.e. we are eventually converging to a local minimum) the Hessian matrix $\nabla^2 f(x_{k+1})$ is positive semidefinite. Furthermore, again the eigenvalues larger than $+1$ in $\Lambda[\nabla^2 f(x_{k+1})]$ are scaled in $\Lambda[M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})]$. Similarly we show in Figures 4.15-4.16 the results for the test function `NONDQUAR` in `CUTEr` collection. The test problems in this optimization framework, where the preconditioner $M_h^\sharp(0,1,I)$ is computed at the outer iteration k and used at the outer iteration $k+1$, confirm that the properties of Theorem 2.1 may hold also when $M_h^\sharp(0,1,I)$ is used on a sequence of linear systems $A_k x = b_k$, when A_k changes *slightly* with k.

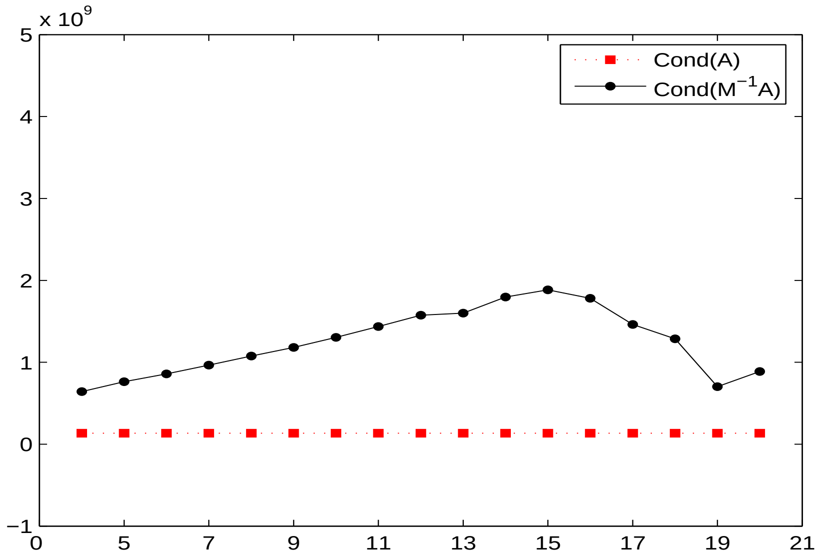

Figure 4.10: The condition number of matrix A_1 ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)A_1$ ($Cond(M^{-1}A)$), when $4 \leq h \leq 20$. The condition number of A_1 is now slightly *larger* than the condition number of $M_h^\sharp(0,1,I)A_1$, for any value of the parameter h. The starting point of the CG is $x_0 = 100e$.

5 Conclusions

We have given theoretical and numerical results for a class of preconditioners, which are parameter dependent. The preconditioners can be built by using any Krylov subspace method for the symmetric linear system (2.1), provided that it is able to satisfy the general conditions (2.2)-(2.3) in Assumption 2.1. The latter property may be appealing in several real problems, where a few iterations of the Krylov subspace method adopted may suffice to compute an effective preconditioner.
Our proposal seems tailored also for those cases where a sequence of linear systems of the form

$$A_k x = b_k, \qquad k = 1, 2, \ldots$$

requires a solution (e.g., see [12] for details), where A_k slightly changes with the index k. In the latter case, the preconditioner $M_h^\sharp(a,\delta,D)$ in (2.4)-(2.5) can be computed applying the Krylov subspace method to the first linear system $A_1 x = b_1$. Then, $M_h^\sharp(a,\delta,D)$ can be used to efficiently solve $A_k x = b_k$, with $k = 2, 3, \ldots$

Finally, the class of preconditioners in this paper seems a promising tool also for the solution of linear systems in financial frameworks. In particular, we want to focus on symmetric linear systems arising when we impose KKT conditions in portfolio optimization problems, with a large number of titles in the portfolio, along with linear equality constraints [2].

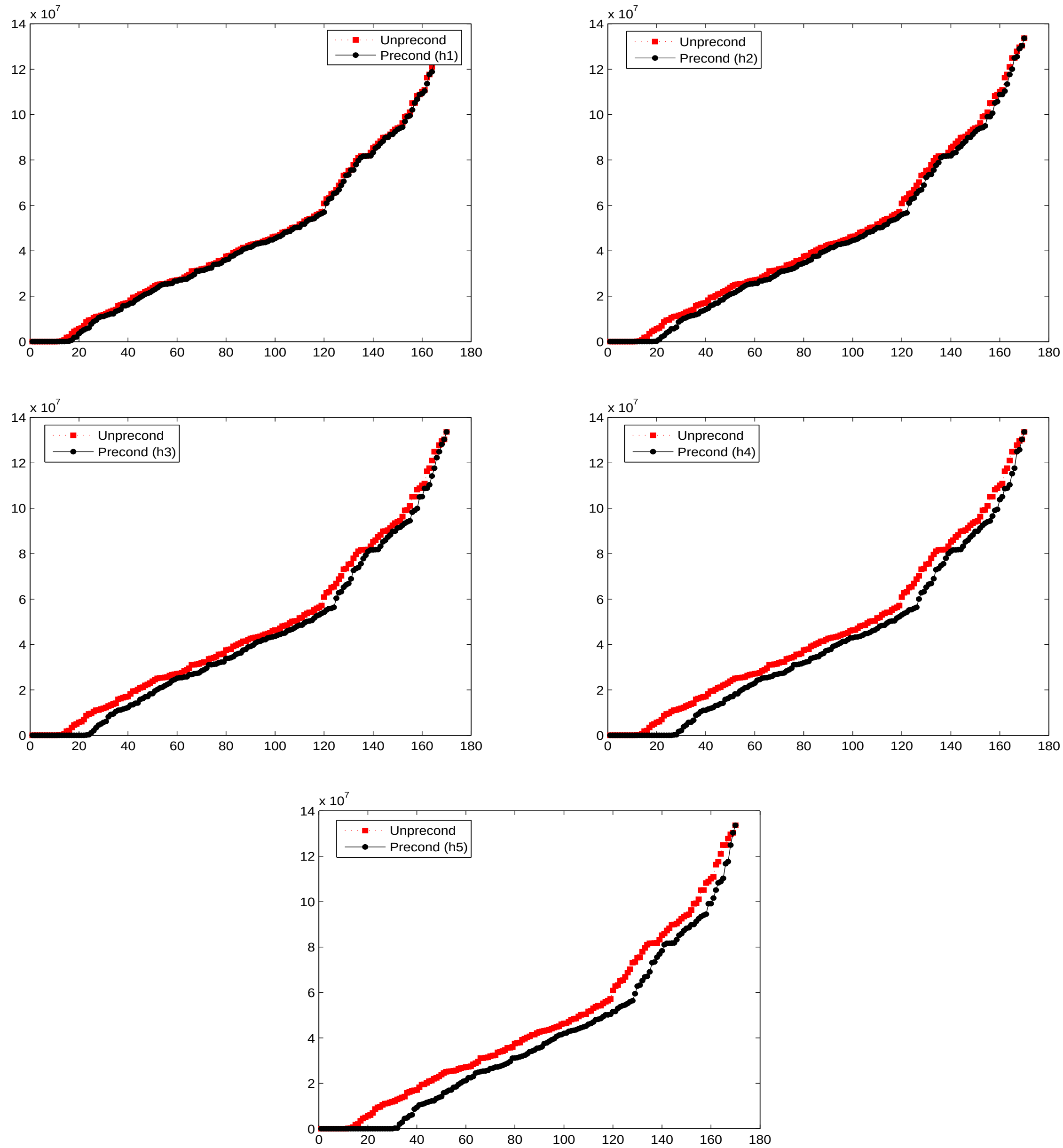

Figure 4.11: Comparison between the *full spectra* $\Lambda[A_1]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A_1]$ (*Precond*); the eigenvalues are sorted for simplicity). We used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$). Again, the eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A_1]$ are in general smaller than the corresponding eigenvalues in $\Lambda[A_1]$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A_1]$ are more clustered near $+1$. The starting point of the CG is $x_0 = 0$.

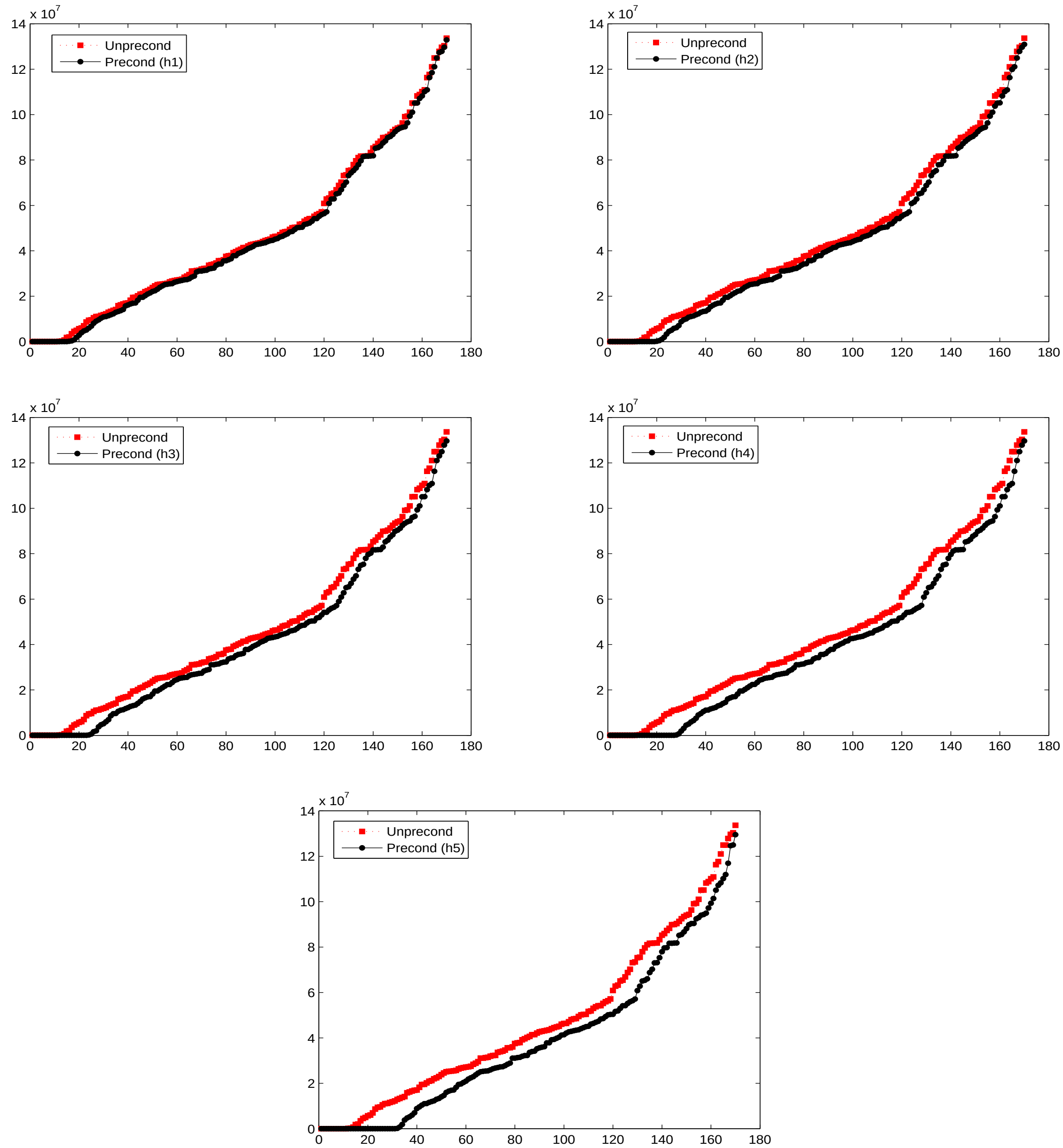

Figure 4.12: Comparison between the *full spectra* $\Lambda[A_1]$ (*Unprecond*) and $\Lambda[M_h^\sharp(0,1,I)A_1]$ (*Precond*); the eigenvalues are sorted for simplicity. We used different values of h ($h1 = 4$, $h2 = 8$, $h3 = 12$, $h4 = 16$, $h5 = 20$). Again, the eigenvalues in the spectrum $\Lambda[M_h^\sharp(0,1,I)A_1]$ are in general smaller than the corresponding eigenvalues in $\Lambda[A_1]$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)A_1]$ are more clustered near $+1$. The starting point of the CG is $x_0 = 100e$.

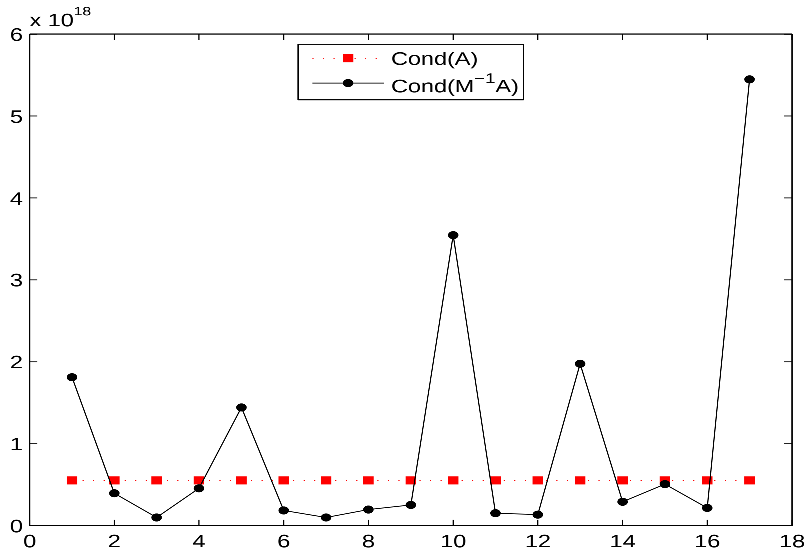

Figure 4.13: The condition number of matrix $\nabla^2 f(x_{k+1})$ $(Cond(A))$ along with the condition number of matrix $M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})$ $(Cond(M^{-1}A))$, for the optimization problem NONCVXUN, when $1 \leq h \leq 17$. The condition number of $\nabla^2 f(x_{k+1})$ is nearby the condition number of $M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})$, for any value of the parameter h. The value $k = 175$ was the first step such that $\|x_{k+1} - x_k\| \leq 10^{-3}\|x_k\|$ (i.e. x_{k+1} and x_k are sufficiently close) and $\alpha_k \geq 0.95$ (i.e. we are likely close to the minimum point). In particular it was $\|x_{175} - x_{176}\| \approx 0.083$.

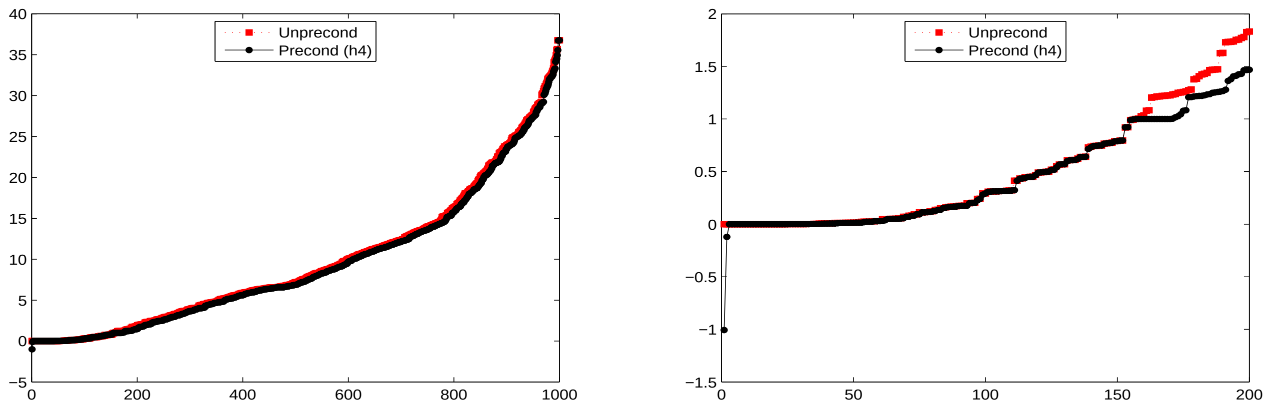

Figure 4.14: Comparison between the *full spectra/detailed spectra* (left figure/right figure) of $\nabla^2 f(x_{k+1})$ (*Unprecond*) and $M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})$ (*Precond*), for the optimization problem NONCVXUN, with $h = h4 = 16$. The eigenvalues in $\Lambda[M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})]$ larger than $+1$ are evidently attenuated, so that $\Lambda[M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})]$ is more clustered.

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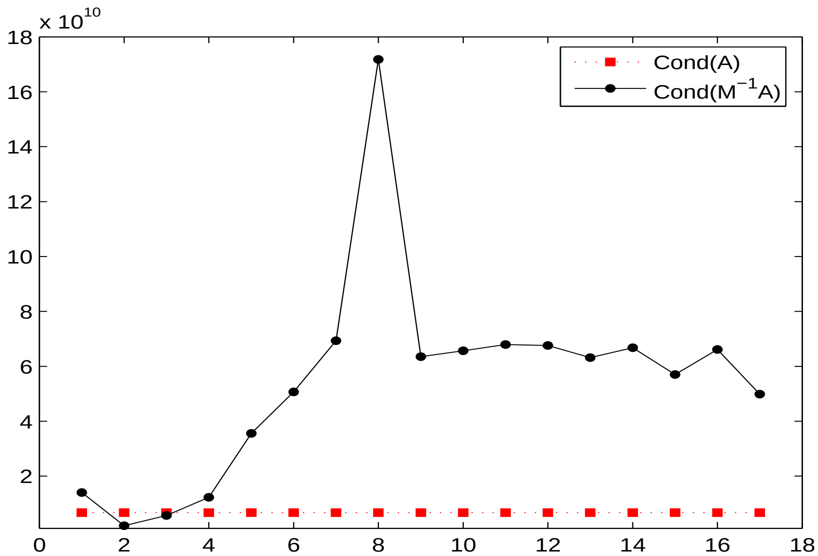

Figure 4.15: The condition number of matrix $\nabla^2 f(x_{k+1})$ ($Cond(A)$) along with the condition number of matrix $M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})$ ($Cond(M^{-1}A)$), for the optimization problem NONDQUAR, when $1 \leq h \leq 17$. The condition number of $\nabla^2 f(x_{k+1})$ is now slightly larger than the condition number of $M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})$ (though they are both $\approx 10^{10}$). The value $k = 40$ was the first step such that $\|x_{k+1} - x_k\| \leq 10^{-3}\|x_k\|$ (i.e. x_{k+1} and x_k are sufficiently close) and $\alpha_k \geq 0.95$ (i.e. we are likely close to the minimum point). In particular it was $\|x_{40} - x_{41}\| \approx 0.203$.

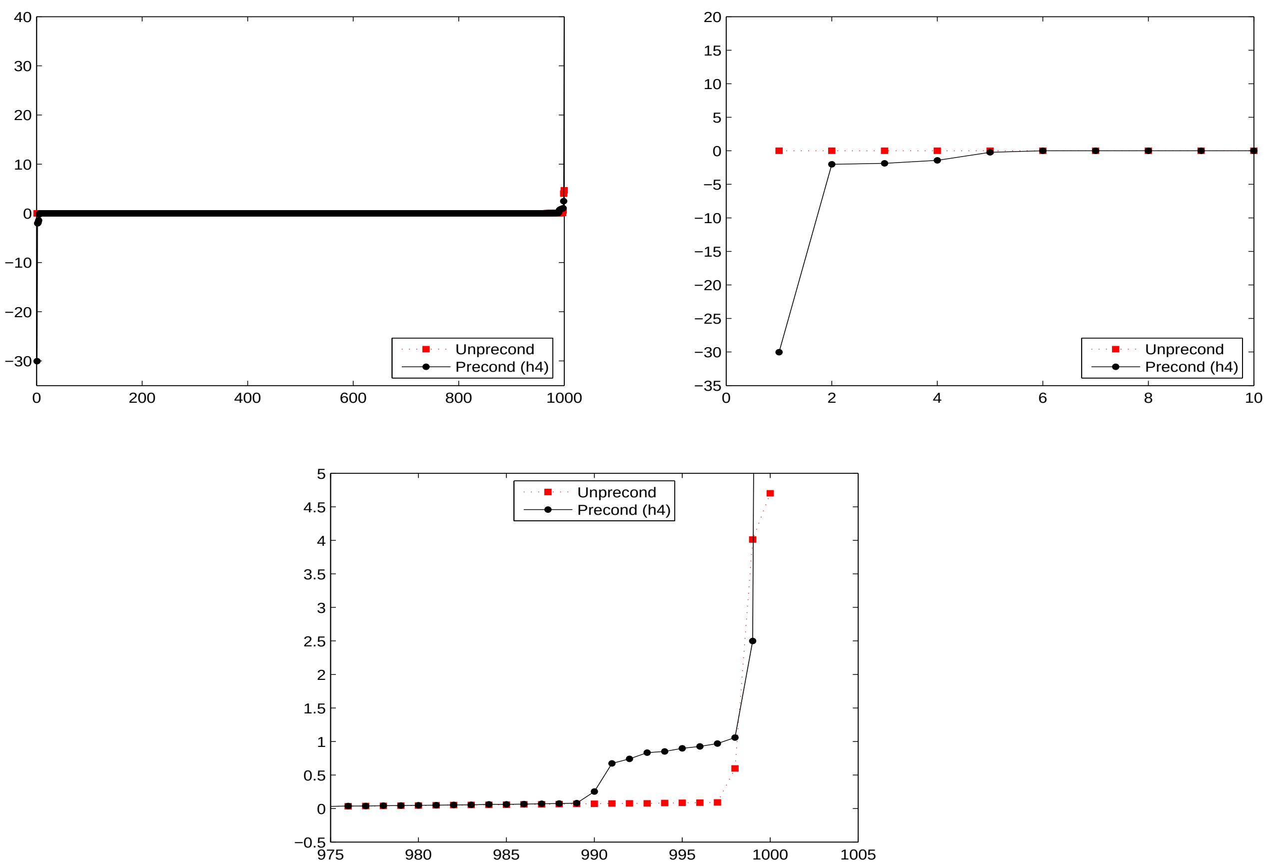

Figure 4.16: Comparison between the *full spectra/detailed spectra* (upper figure/lower figures) $\Lambda[\nabla^2 f(x_{k+1})]$ (*Unprecond*) and $\Lambda[M_h^{-1}\nabla^2 f(x_{k+1})]$ (*Precond*), for the optimization problem NONDQUAR, with $h = h4 = 16$. Some nearly-zero eigenvalues in the spectrum $\Lambda[\nabla^2 f(x_{k+1})]$ are shifted to non-zero values in $\Lambda[M_h^\sharp(0,1,I)\nabla^2 f(x_{k+1})]$. Since many eigenvalues in $\Lambda[\nabla^2 f(x_{k+1})]$ are zero or nearly-zero, the preconditioner $M_h^\sharp(0,1,I)$ may be of scarce effect, unless large values of the parameter h are considered.

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