Stochastic model of agent interaction with opinion leaders

Andrea Ellero,[1,*] Giovanni Fasano,[1,2,†] and Annamaria Sorato[1,‡]

[1]*Department of Management, Ca' Foscari University of Venice, Venice, Italy*
[2]*INSEAN-CNR Italian Ship Model Basin, Rome, Italy*

We analyze the problem of agents' interactions in a given population. The purpose of this paper is twofold. Starting from a scheme proposed by Galam [Physica A **320**, 571 (2003)], which is based on a majority rule to treat the individuals' interactions, we first study some of its relevant properties. Then, we introduce special individuals, called *opinion leaders*, who play a key role in information spreading in several practical applications. Opinion leaders have the special feature of strongly interfering with the process based on the majority rule, speeding up the diffusion. We consider a model describing agents' interactions, which encompasses Galam's proposal, where opinion leaders are included as special agents. Then we study its specific properties which significantly recast and extend some conclusions drawn for the models given by Galam and Ellero, Fasano, and Sorato [Physica A **388**, 3901 (2009)]. Finally, we provide theoretical and numerical results concerning the dynamics of our model, showing that a small percentage of opinion leaders may both accelerate and/or even reverse the overall consensus among all the agents.

PACS number(s): 89.65.−s, 02.50.Ey

I. INTRODUCTION

Diffusion dynamics applied to social sciences has been studied in a huge number of papers and books, covering a wide range of perspectives, from marketing (see, e.g., Refs. [1–3]) to agent-based modeling (see, e.g., Refs. [4,5]) and sociophysics (see, e.g., Refs. [6,7]). As originally pointed out in Ref. [8], some members of a social network are "likely to influence other persons in their immediate environment" and enhance diffusion processes. They affect the spreading of information or the diffusion and adoption of a new product through different communication channels [9]. In the literature, such special people are called *opinion leaders*, *influencers*, *mavens*, or *hubs*, depending on the kind of influence on the process they have, and are usually convincing experts or have a large number of social ties [10–13].

The interest in opinion leaders has been increasing in recent years due to the rapid spread of social networks and the Web 2.0, where rather easily a small group of opinion leaders may accelerate, or even stop, information circulation, especially in the initial phase of the process [12,14]. As a result, influencer marketing is nowadays a recognized form of communication, which requires a specific toolbox to be fully exploited [15].

In this paper we focus on opinion leaders who are effective in modifying the dynamics of individuals' interactions due to their influencing skills, while their connectedness in the social network plays a secondary role. As a practical example of this kind of leader, consider the way some influential doctors are able to push products of specific drug companies. Pharmaceutical firms reward opinion leaders among doctors to turn them into disguised salespersons [16], boosting in this way the process of the drug's diffusion on the doctors' social network [11].

The interaction of politicians with electorates provides another real problem where opinion leaders may play a key role, in order to spread information and to influence final decisions. One of the early examples in this regard is given in Ref. [17].

Our idea of opinion leaders has also some relation with the concept of *inflexibles*, special individuals introduced by Galam [18], who never change their minds about some issues. On the other hand, opinion leaders have a stronger and more active role in the communication process with respect to inflexibles, who have the effect of adding some inertia to the diffusion process.

The underlying diffusion process we consider in this paper relies on a stylized model proposed by Galam [6]. In Galam's model each member (agent) of an undifferentiated population can have one of two opposite opinions about some topic, and may change her mind in discussions with other individuals of the population. Thus, each individual may impact opinion diffusion by means of repeated discussions in groups.

In our proposal we suitably recast Galam's model, in order to introduce opinion-leader agents, i.e., differentiated agents who are assumed to drive the opinion of the agents joining the same discussion group. The opinion leaders we consider are able to convince all the agents they meet in a group, while keeping their own opinions unchanged. We provide both theoretical and numerical results which show the role of opinion leaders in the diffusion process.

The paper is organized as follows: Sec. II partially reviews Galam's model [6], detailing some of its peculiarities. Section III is devoted to reporting some theoretical properties of Galam's model. In Sec. IV we describe our model, including the definition of opinion leaders. Section V analyzes the dynamics of the model with opinion leaders, also comparing it with Galam's model. Finally, a section of Conclusions completes the paper, along with an Appendix of additional results for relevant special cases.

In order to preserve fluent readability for uninterested readers, all the proofs of theoretical results have been moved to the Appendix. This helps both preserve a rigorous treatment of the subject and allow a fast comprehension of the paper's contents.

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*ellero@unive.it
†fasano@unive.it
‡amsorato@unive.it

II. GALAM's MODEL

This section briefly reviews Galam's model [6,7] along with some of its properties. Consider a set of N individuals (agents) involved in a discussion, who may take one of two opposite opinions (say, "+" or "−") about a certain topic. At each time step t (with $t \geqslant 0$) the N agents meet to discuss. Let $N_+(t) [N_-(t)]$ be the number of agents having opinion $+ (-)$; obviously $N = N_+(t) + N_-(t)$ at each time step t. When the agents meet, the discussion takes place in subsets (groups) of individuals, and they may change their minds.

More precisely, at time t, each of the N agents belongs to a k-sized group with probability a_k, where k denotes the cardinality of the group, $k = 1, \ldots, L$. Thus, at each time step there may be several groups of size k and

$$\sum_{k=1}^{L} a_k = 1 \quad \text{with} \quad a_k \geqslant 0, \quad k = 1, \ldots, L.$$

Then, after a discussion in each group, at the outset of the next period $t + 1$, each individual can reverse their opinion to the opposite one ($+$ becomes $-$ or vice versa) according

to a majority rule, i.e., all agents of the group take the view of the majority. The rule for reversing opinion is assumed by Galam [6] to be slightly biased in favor of one of the two opinions, namely, the negative opinion $-$: in the case of parity the opinion $-$ prevails over $+$, so that all the members of the group will take opinion $-$ at time $t + 1$.

Let us indicate by $P_+(t)$ the "estimated" probability that an individual, among the N individuals of the population, thinks $+$ at time t. Clearly the probability that an individual thinks $-$ at step t must be given by $P_-(t) = 1 - P_+(t)$.

Galam [6] provides a model to be used to estimate the dynamics of the probability $P_+(t)$ using the recursive formula

$$P_+(t + 1) = \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}, \quad (1)$$

where the symbol $\lfloor z \rfloor$ indicates the largest integer less than or equal to z, and C_j^k is the binomial coefficient. In order to explicitly consider the cases in which $P_+(t)$ is equal to 0 or 1 we extend (1) as follows:

$$P_+(t + 1) = \mathcal{G}[P_+(t)] = \begin{cases} 0 & \text{if } P_+(t) = 0, \\ \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j} & \text{if } 0 < P_+(t) < 1, \\ 1 & \text{if } P_+(t) = 1, \end{cases} \quad (2)$$

which defines the continuous function $\mathcal{G}$ on [0,1]. We set the initial condition $P_+(0) = N_+(0)/N$, where $N_+(0)$ is the number of agents thinking $+$ at the beginning of the process. For any $t \geqslant 1$ the quantity $P_+(t)$ is computed by (2), and may possibly differ from the "actual" value $N_+(t)/N$ (which represents the probability of having individuals thinking $+$ at time t, among the N individuals).

Note that model (2) is well posed and the quantity $P_+(t)$ satisfies $0 \leqslant P_+(t) \leqslant 1$, for any $t \geqslant 1$. Indeed, except for the trivial cases in which $P_+(t)$ is 0 or 1, in (2) for any choice of $a_1, \ldots, a_L$ we have $P_+(t + 1) < 1$ if

$$\sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j} < 1,$$

for any $k = 1, \ldots, L$. The latter inequality holds from the binomial theorem, since

$$\sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}$$

$$< \sum_{j=0}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}$$

$$= \{P_+(t) + [1 - P_+(t)]\}^k = 1.$$

Following Refs. [6,19,20] we define the *killing point* (or *tipping point*) as the threshold value $\hat{P}_+$ such that

    (a) if $P_+(0) > \hat{P}_+$ then $\lim_{t \to \infty} P_+(t) = 1$,

    (b) if $P_+(0) = \hat{P}_+$ then $P_+(t) = P_+(0)$, $\forall \, t > 0$,

    (c) if $P_+(0) < \hat{P}_+$ then $\lim_{t \to \infty} P_+(t) = 0$.

In other words, the killing point $\hat{P}_+$ is an unstable fixed point of the function $\mathcal{G}$ and also a threshold value such that, if the probability of agents thinking $+$ at the beginning of the process lies above it, then all agents will definitely have opinion $+$. Conversely, the opinion of the agents will be definitely $-$ if the starting probability of agents thinking $+$ is less than $\hat{P}_+$.

We complete this section with a reformulation of (1), in a way that will be useful when we recast Galam's model in Sec. IV. Observe that from the relation

$$\left\lfloor \frac{k}{2} + 1 \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor + 1,$$

and recalling that for the binomial distribution

$$\sum_{j=0}^{k} C_j^k \{P_+(t)\}^j \{1 - P_+(t)\}^{k-j} = 1,$$

we have

$$\sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}$$

$$= 1 - \sum_{k=1}^{L} a_k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}.$$

Model (1) is therefore equivalent to

$$P_+(t + 1) = 1 - \sum_{k=1}^{L} a_k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}. \quad (3)$$

III. SOME TECHNICAL PROPERTIES OF GALAM's MODEL

In this section we consider some properties of the dynamics of model (1). Most of the results in this section might be rather technical for nonspecialist readers. However, we are going to show that this preliminary analysis is useful, in order to provide additional results relative to both Galam's model and its extension proposed in Sec. IV. To facilitate reading at first glance all the proofs are reported in the Appendix. Some extensions to the case when opinion leaders are taken into account in the diffusion process are detailed in Sec. V.

Recalling (2), let us indicate by $\mathcal{G}_k$, with $1 \leqslant k \leqslant L$, the function

$$
\mathcal{G}_k[P_+(t)]
= \begin{cases}
0 & \text{if } P_+(t) = 0, \\
\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j} & \text{if } 0 < P_+(t) < 1, \\
1 & \text{if } P_+(t) = 1,
\end{cases}
\tag{4}
$$

which is model (2) with $a_j = 0$, for all $j \neq k$. Then we immediately realize that $\mathcal{G}$ in (2) is given by the weighted sum

$$
\mathcal{G} = \sum_{k=1}^{L} a_k \mathcal{G}_k. \tag{5}
$$

Relation (5) suggests that in order to study some general properties of Galam's model (2), it might be useful to consider what happens in the special cases in which all the groups have the same size, i.e., we consider $a_k = 1$ for some fixed value of k.

With this view, let us rewrite the function $\mathcal{G}_k$. Following Ref. [21] we have

$$
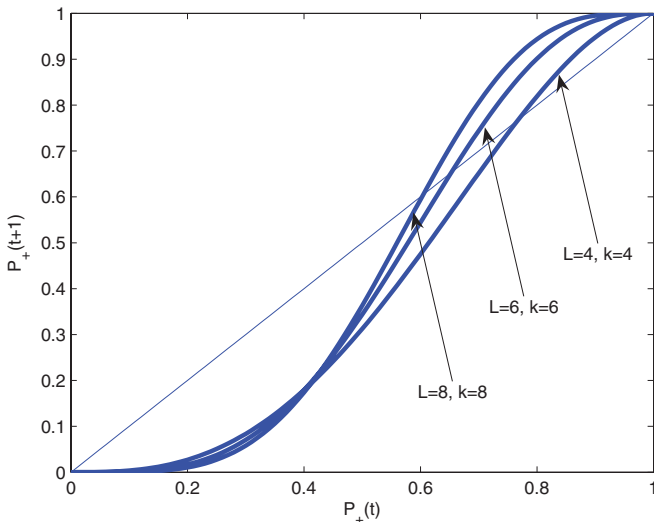\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}
$$

FIG. 1. (Color online) $P_+(t+1)$ vs $P_+(t)$ as given by Galam's formula (1) with $a_k = 1$, and $k \in \{4,6,8\}$. Since k is even, in all the three cases the killing point is larger than 0.5.

$$
= \frac{B_C \left(P_+(t); \lfloor \frac{k}{2} \rfloor + 1, k - \lfloor \frac{k}{2} \rfloor \right)}{B \left(\lfloor \frac{k}{2} \rfloor + 1, k - \lfloor \frac{k}{2} \rfloor \right)}, \tag{6}
$$

where

$$
B_C \left(P_+(t); \left\lfloor \frac{k}{2} \right\rfloor + 1, k - \left\lfloor \frac{k}{2} \right\rfloor \right)
$$
$$
= \int_0^{P_+(t)} u^{\lfloor \frac{k}{2} \rfloor} (1 - u)^{k - \lfloor \frac{k}{2} \rfloor - 1} du \tag{7}
$$

is the so-called *complete beta function*, and

$$
B \left(\left\lfloor \frac{k}{2} \right\rfloor + 1, k - \left\lfloor \frac{k}{2} \right\rfloor \right) = \frac{\lfloor \frac{k}{2} \rfloor! \, (k - \lfloor \frac{k}{2} \rfloor - 1)!}{k!} \tag{8}
$$

is the *beta function*. In order to simplify the notation in the following we define

$$
\beta_k = \frac{\lfloor \frac{k}{2} \rfloor! \, (k - \lfloor \frac{k}{2} \rfloor - 1)!}{k!}.
$$

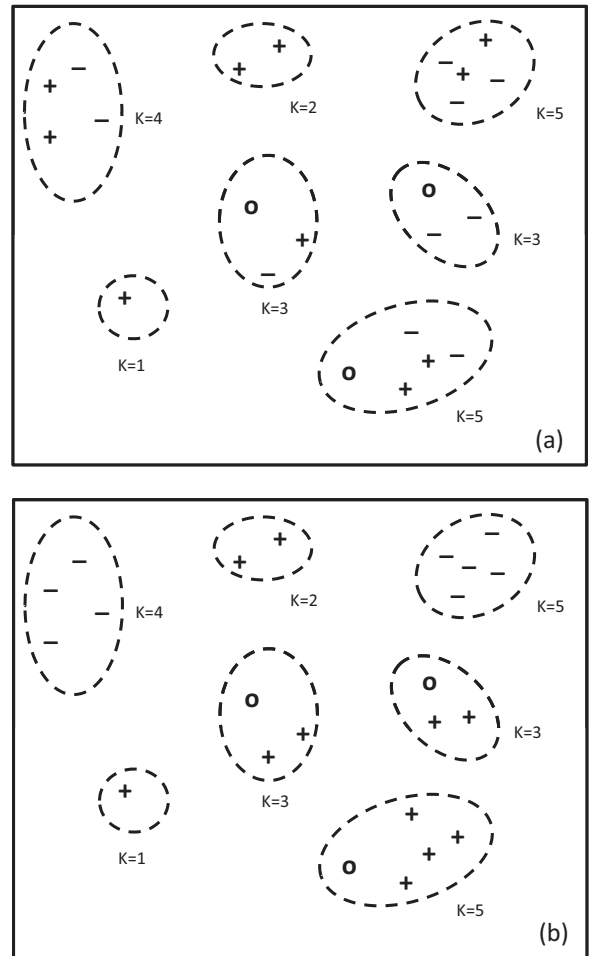Using (6)–(8) we have the following result.

FIG. 2. A possible transformation of the groups of agents, in the presence of opinion leaders, from the outset of time step t (a) to the end of time step t (b), after discussion. With $+$ we indicate agents thinking $+$, with $-$ agents thinking $-$, and o represents an opinion leader. The agents are grouped into tables of discussion whose size is indicated by K.

TABLE I. The killing point $\hat{P}_+$ of $\mathcal{G}_k$, for even k and $2 \leqslant k \leqslant 50$.

$k=2$	$\hat{P}_+ = 1.0000$	$k=4$	$\hat{P}_+ = 0.7676$	$k=6$	$\hat{P}_+ = 0.6529$	$k=8$	$\hat{P}_+ = 0.6045$	$k=10$	$\hat{P}_+ = 0.5786$
$k=12$	$\hat{P}_+ = 0.5626$	$k=14$	$\hat{P}_+ = 0.5519$	$k=16$	$\hat{P}_+ = 0.5442$	$k=18$	$\hat{P}_+ = 0.5384$	$k=20$	$\hat{P}_+ = 0.5340$
$k=22$	$\hat{P}_+ = 0.5304$	$k=24$	$\hat{P}_+ = 0.5275$	$k=26$	$\hat{P}_+ = 0.5251$	$k=28$	$\hat{P}_+ = 0.5231$	$k=30$	$\hat{P}_+ = 0.5213$
$k=32$	$\hat{P}_+ = 0.5198$	$k=34$	$\hat{P}_+ = 0.5185$	$k=36$	$\hat{P}_+ = 0.5174$	$k=38$	$\hat{P}_+ = 0.5163$	$k=40$	$\hat{P}_+ = 0.5154$
$k=42$	$\hat{P}_+ = 0.5146$	$k=44$	$\hat{P}_+ = 0.5139$	$k=46$	$\hat{P}_+ = 0.5132$	$k=48$	$\hat{P}_+ = 0.5126$	$k=50$	$\hat{P}_+ = 0.5121$

Proposition 3.1. Consider the function $\mathcal{G}_k$ in (4) for a fixed value of $k \geqslant 1$. Then,

(1) $\mathcal{G}_k : [0,1] \rightarrow [0,1]$.

(2) $\mathcal{G}_k$ is continuous in $[0,1]$ and strictly increasing on $(0,1)$.

(3) For $k = 1$ the function $\mathcal{G}_k$ is the identity.

(4) For k even and $k \geqslant 2$ the function $\mathcal{G}_k$ is strictly convex if and only if $P_+(t) \leqslant k/[2(k-1)]$.

(5) For k odd and $k \geqslant 3$ the function $\mathcal{G}_k$ is strictly convex if and only if $P_+(t) \leqslant 1/2$.

We then have also the next corollary.

Corollary 3.1. If $k \neq 1$ then $\mathcal{G}_k$ has a unique fixed point $\hat{P}_+ \in (0,1)$, which is the killing point of $\mathcal{G}_k$.

Considering relation (5), some properties of function $\mathcal{G}_k$ can be extended to function $\mathcal{G}$.

Corollary 3.2. Let us consider the function $\mathcal{G}$ in (5). Then

(i) $\mathcal{G} : [0,1] \rightarrow [0,1]$.

(ii) $\mathcal{G}$ is continuous in $[0,1]$ and strictly increasing on $(0,1)$.

(iii) If $a_1 = 1$ the function $\mathcal{G}$ is the identity.

(iv) If $a_1 \neq 1$ then $\mathcal{G}$ is strictly convex on $[0,1/2]$.

(v) If $a_1 \neq 1$ and $a_k = 0$ for any even k, then $\mathcal{G}$ is strictly concave on $[1/2,1]$.

(vi) If $a_1 \neq 1$ and $a_k = 0$ for any k odd, then $\mathcal{G}$ is strictly convex on $\{0,\bar{k}/[2(\bar{k}-1)]\}$ where $\bar{k} = \max_{2 \leqslant k \leqslant L}\{k : a_k \neq 0$, and k is even$\}$.

As regards the existence of fixed points for function $\mathcal{G}$, we have the next result.

Corollary 3.3. Consider the function $\mathcal{G}$ in (5). $\mathcal{G}$ has at least one fixed point in $(0,1)$.

When gathering only in odd-sized groups is allowed, then we have the following special property.

Corollary 3.4. If $a_k = 0$ for any even value of $k, 2 \leqslant k \leqslant L$, and $a_1 < 1$, then $\hat{P}_+ = 1/2$ is the killing point of function $\mathcal{G}$.

For a better understanding of the Corollary 3.4, in Fig. 4(a) we report the graph of $\mathcal{G}_k$, i.e., the scatter plot graphs of $P_+(t+1)$ vs $P_+(t)$, choosing $k \in \{5,7,9\}$ (i.e., for some *odd* values

TABLE II. Comparison between the model (10) and a simulation, where we set respectively $L = 6$, $a_k = 1/6$, $k = 1, \ldots, 6$ or $L = 7$, $a_k = 1/7$, $k = 1, \ldots, 7$, and $s \in \{0, 0.06\}$ (similar results hold also for other values of s). The value $\sigma^2(t)$ represents, at time step t (t step), the average square error between the model (10) and the simulation, over 500 runs. The simulation is performed setting $N = 1000$; the results are reported for ten steps of the simulation. We set $N_+(0) = 450$ so that the initial percentage of $+$ agents [i.e., $P_+(0) = 450/1000$] was below the killing point computed for $s = 0$. Finally, the results are averaged over 500 runs. The model and the simulation provide similar results, for both $L = 6$ and $L = 7$. The numbers in square brackets indicate the power of 10.

	$L=6,\quad s=0$				$L=6,\quad s=0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3302	0.3298	$0.670 - 07$	1	0.4229	0.4230	$0.390 - 06$
2	0.1842	0.1851	$0.572 - 06$	2	0.3943	0.3945	$0.313 - 06$
3	0.0647	0.0660	$0.969 - 07$	3	0.3659	0.3666	$0.607 - 06$
4	0.0141	0.0145	$0.606 - 07$	4	0.3397	0.3420	$0.499 - 07$
5	0.0025	0.0026	$0.138 - 07$	5	0.3174	0.3214	$0.550 - 06$
6	0.0004	0.0004	$0.383 - 09$	6	0.3000	0.3034	$0.485 - 06$
7	0.0001	0.0001	$0.147 - 10$	7	0.2873	0.2882	$0.929 - 08$
8	0.0000	0.0000	$0.288 - 12$	8	0.2786	0.2792	$0.207 - 06$
9	0.0000	0.0000	$0.320 - 13$	9	0.2728	0.2716	$0.110 - 06$
	$L=7,\quad s=0$				$L=7,\quad s=0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3390	0.3388	$0.664 - 06$	1	0.4423	0.4400	$0.580 - 06$
2	0.1919	0.1910	$0.114 - 05$	2	0.4337	0.4301	$0.144 - 05$
3	0.0635	0.0647	$0.820 - 06$	3	0.4242	0.4212	$0.202 - 05$
4	0.0119	0.0123	$0.227 - 06$	4	0.4139	0.4115	$0.110 - 05$
5	0.0018	0.0019	$0.239 - 08$	5	0.4031	0.4037	$0.138 - 06$
6	0.0003	0.0003	$0.104 - 08$	6	0.3920	0.3970	$0.320 - 05$
7	0.0000	0.0000	$0.156 - 11$	7	0.3809	0.3886	$0.706 - 05$
8	0.0000	0.0000	$0.320 - 13$	8	0.3702	0.3817	$0.656 - 05$
9	0.0000	0.0000	$0.800 - 14$	9	0.3603	0.3764	$0.656 - 05$

of k). The resulting graphs are summarized in Fig. 4(a), and confirm Corollary 3.4.

Observe that Corollary 3.4 does not hold in the case $a_k = 0$ for any odd value of k, due to the bias for $-$ in the case of parity. Similarly, Corollary 3.4 does not hold in the case $a_k \neq 0$, for at least one even k. An example is given in Fig. 1, where we set $a_k = 1$ with $k \in \{4,6,8\}$. Observe that due to the bias of the majority rule in favor of $-$, the killing point when k is even cannot be smaller than $1/2$. In particular, as Fig. 1 suggests and in view of (5), we are able to provide lower and upper bounds for the killing point in the general case. Indeed, the value of the killing point of $\mathcal{G}$ is bounded from below by $1/2$, as mentioned before, and bounded from above by the killing point of $\mathcal{G}_m$, where

$$m = \min_{2 \leqslant k \leqslant L} \{k : k \text{ is even and } a_k \neq 0\}. \quad (9)$$

Since the killing point of $\mathcal{G}_4$ is $(1 + \sqrt{13})/6 \approx 0.768$, then the latter value is an upper bound for the killing point of $\mathcal{G}$, provided that $a_2 = 0$.[1] We report in Table I the killing point of $\mathcal{G}_k$, for even values of k in the range [2,50]. Table I confirms the bound (9), since the killing point $\hat{P}_+$ decreases as k increases.

In Appendix we explicitly compute the killing point of $\mathcal{G}$ for the special cases $L = 1,2,3,4$ (see also Ref. [19]).

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[1] In Appendix we prove that when $L = 2$ then $\hat{P}_+ = 1$.

From Fig. 4(a) and Fig. 1 there is also empirical evidence that, given the value of probability $\bar{P}_+ > 1/2$, then $\mathcal{G}_k(\bar{P}_+)$ increases with k (in Sec. V we address the latter issue more precisely).

IV. A PERSPECTIVE INTRODUCING OPINION LEADERS

We consider now the scenario that appears as a result of the introduction, in the process defined by Galam [6], of special agents which are opinion leaders and, in particular, we focus on their effects on the opinion diffusion dynamics. To clarify our approach we introduce the following definition of opinion leader.

Definition IV.1. Given N agents, an agent is called an opinion leader if

(1) Her opinion is $+$, for any $t \geqslant 0$.

(2) All agents in the group she is joining at time step t will have opinion $+$ at the beginning of time step $t + 1$.

The first requirement in the above definition means that each opinion leader is one of the $N_+(t)$ agents who think $+$ at each time step t, while the second requirement assumes that an opinion leader is able to *convince* all the members of her group to think $+$, no matter the opinion they had before gathering.

Thus, from Definition 4.1 the role of the opinion leader we have just introduced, to a large extent summarizes some informal definitions given in the literature (see, e.g.,

TABLE III. Comparison between the model (10) and a simulation, where we set respectively $L = 6$, $a_k = 1/6$, $k = 1, \ldots, 6$ or $L = 7$, $a_k = 1/7$, $k = 1, \ldots, 7$, and $s \in \{0, 0.06\}$ (similar results hold also for other values of s). The value $\sigma^2(t)$ represents, at time step t, the average square error between the model (10) and the simulation, over 500 runs. The simulation is performed setting $N = 700$; the results are reported for ten steps of the simulation. We set $N_+(0) = 315$ so that the initial percentage of $+$ agents [i.e., $P_+(0) = 315/700$] was below the killing point computed for $s = 0$. Finally, the results are averaged over 500 runs. The model and the simulation provide similar results, both for $L = 6$ and $L = 7$. The numbers in square brackets indicate the power of 10.

	$L = 6,\ s = 0$				$L = 6,\ s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.0000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3302	0.3308	$0.173 - 06$	1	0.4229	0.4224	$0.161 - 06$
2	0.1842	0.1847	$0.114 - 06$	2	0.3943	0.3938	$0.305 - 06$
3	0.0647	0.0659	$0.106 - 06$	3	0.3659	0.3669	$0.177 - 05$
4	0.0141	0.0149	$0.699 - 09$	4	0.3397	0.3417	$0.239 - 05$
5	0.0025	0.0027	$0.184 - 07$	5	0.3174	0.3205	$0.222 - 05$
6	0.0004	0.0005	$0.190 - 08$	6	0.3000	0.3033	$0.136 - 05$
7	0.0001	0.0001	$0.110 - 10$	7	0.2873	0.2915	$0.375 - 07$
8	0.0000	0.0000	$0.261 - 12$	8	0.2786	0.2831	$0.139 - 07$
9	0.0000	0.0000	$0.163 - 13$	9	0.2728	0.2753	$0.833 - 06$
	$L = 7,\ s = 0$				$L = 7,\ s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3390	0.3379	$0.295 - 06$	1	0.4423	0.4409	$0.1300 - 06$
2	0.1919	0.1907	$0.107 - 08$	2	0.4337	0.4323	$0.854 - 07$
3	0.0635	0.0644	$0.341 - 07$	3	0.4242	0.4235	$0.512 - 05$
4	0.0119	0.0129	$0.363 - 08$	4	0.4139	0.4133	$0.153 - 04$
5	0.0018	0.0020	$0.659 - 09$	5	0.4031	0.4072	$0.368 - 04$
6	0.0003	0.0003	$0.132 - 09$	6	0.3920	0.4011	$0.300 - 04$
7	0.0000	0.0000	$0.163 - 11$	7	0.3809	0.3959	$0.255 - 04$
8	0.0000	0.0000	$0.163 - 13$	8	0.3702	0.3907	$0.130 - 04$
9	0.0000	0.0000	$0.000 + 00$	9	0.3603	0.3856	$0.650 - 05$

Refs. [8,10,11]). Moreover, the concept of the opinion leader has some analogy with the idea of inflexible agent defined in Ref. [18].

Following the ideas of Galam's model we want to estimate the probability $P_+(t+1)$ of having individuals thinking $+$ at period $t+1$ when opinion leaders take part in the diffusion process. Defining, as for (1), the parameters $L, a_1, \ldots, a_L$, and the probability $P_+(t)$, we consider a process in which exactly N_{op} opinion leaders are activated, clearly with $N_{\mathrm{op}} \leqslant N_+(t)$, for any t. Keeping the behavior biased toward the opinion $-$ in the case of parity, we propose the following *opinion leader model* which extends Galam's idea as expressed in (3):

$$P_+(t+1) = 1 - \sum_{k=1}^{L} a_k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} C_j^k \{P_+(t) - s\}^j \{1 - P_+(t)\}^{k-j},$$

(10)

where $s = N_{\mathrm{op}}/N$ is the probability for an agent to be an opinion leader. Note that s is independent of t and, obviously, since by definition each opinion leader thinks $+$, we must have $s \leqslant P_+(t) \leqslant 1$. In addition, the introduction of s encompasses also those cases where possibly both $N \to \infty$ (in Definition 4.1) and $N_{\mathrm{op}} \to \infty$, but N_{op}/N is finite.

Observe that the quantity

$$C_j^k \{P_+(t) - s\}^j \{1 - P_+(t)\}^{k-j} \qquad (11)$$

represents the probability that, at time t, in a k-sized group, exactly j agents think $+$ without being opinion leaders and the remaining $k - j$ agents think $-$. Thus, the sum

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} C_j^k \{P_+(t) - s\}^j \{1 - P_+(t)\}^{k-j},$$

which appears in (10), approximates the probability that an individual in a k-sized group will think $-$ at the end of the period t of the diffusion process.

We also observe that the probability (11) is a special case of a more general trinomial distribution, since it implicitly considers three independent events: "the agent thinks $+$ without being an opinion leader", "the agent thinks $-$," and "the agent is an opinion leader". The probability to have, in a group of size k, exactly j agents (not opinion leaders) thinking $+$, h opinion leaders, and $k - j - h$ agents thinking $-$, is given by

$$C_j^k C_h^{k-j} \{P_+(t) - s\}^j (s)^h \{1 - P_+(t)\}^{k-j-h},$$

which indeed reduces to (11) when $h = 0$, i.e., in the k-sized group there are no opinion leaders. The latter formula suggests possible extensions of the model.

In order to clarify the behavior of opinion leaders in the diffusion process, we report in Fig. 2 a possible transformation of the groups of agents, in the presence of opinion leaders, from time step t to time step $t+1$.

TABLE IV. Comparison between the model (10) and a simulation, where we set respectively $L = 6$, $a_k = 1/6$, $k = 1, \ldots, 6$ or $L = 7$, $a_k = 1/7$, $k = 1, \ldots, 7$, and $s \in \{0, 0.06\}$ (similar results hold also for other values of s). The value $\sigma^2(t)$ represents, at time step t, the average square error between the model (10) and the simulation, over 500 runs. The simulation is performed setting $N = 400$; the results are reported for 10 steps of the simulation. We set $N_+(0) = 180$ so that the initial percentage of $+$ agents [i.e., $P_+(0) = 180/400$] was below the killing point computed for $s = 0$. Finally, the results are averaged over 500 runs. The model and the simulation provide similar results, for both $L = 6$ and $L = 7$. The numbers in square brackets indicate the power of 10.

	$L = 6, \; s = 0$				$L = 6, \; s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3302	0.3294	$0.192 - 07$	1	0.4229	0.4217	$0.168 - 06$
2	0.1842	0.1839	$0.258 - 06$	2	0.3943	0.3905	$0.842 - 06$
3	0.0647	0.0656	$0.852 - 06$	3	0.3659	0.3647	$0.356 - 05$
4	0.0141	0.0156	$0.186 - 07$	4	0.3397	0.3407	$0.219 - 05$
5	0.0025	0.0028	$0.981 - 08$	5	0.3174	0.3204	$0.831 - 06$
6	0.0004	0.0004	$0.361 - 09$	6	0.3000	0.3057	$0.999 - 05$
7	0.0001	0.0000	$0.500 - 11$	7	0.2873	0.2942	$0.761 - 05$
8	0.0000	0.0000	$0.000 + 00$	8	0.2786	0.2857	$0.799 - 05$
9	0.0000	0.0000	$0.000 + 00$	9	0.2728	0.2800	$0.180 - 05$
	$L = 7, \; s = 0$				$L = 7, \; s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3390	0.3383	$0.123 - 08$	1	0.4423	0.4410	$0.725 - 06$
2	0.1919	0.1942	$0.402 - 06$	2	0.4337	0.4309	$0.115 - 05$
3	0.0635	0.0677	$0.126 - 05$	3	0.4242	0.4216	$0.360 - 06$
4	0.0119	0.0141	$0.881 - 07$	4	0.4139	0.4120	$0.128 - 06$
5	0.0018	0.0023	$0.105 - 07$	5	0.4031	0.4062	$0.380 - 06$
6	0.0003	0.0004	$0.245 - 09$	6	0.3920	0.4025	$0.361 - 05$
7	0.0000	0.0000	$0.245 - 11$	7	0.3809	0.3982	$0.644 - 05$
8	0.0000	0.0000	$0.450 - 12$	8	0.3702	0.3953	$0.277 - 05$
9	0.0000	0.0000	$0.000 + 00$	9	0.3603	0.3950	$0.311 - 06$

Furthermore, we have carried out a numerical comparison between the model (10) and a simulation of the opinion dynamics among agents (see also Ref. [20]). The simulation is first performed considering a population of $N \in \{100, 400, 700, 1000\}$ agents (Tables II–V), and considering $P_+(0) = 0.45$. This means that the initial percentage of $+$ agents turns out to be below the killing point computed for $s = 0$. The overall numerical experience included the computation of ten steps of the simulation and the results were averaged over 500 runs. Finally, the model and the simulation were considered for both $L = 6$ and $L = 7$, with respectively $a_k = 1/6$, $k = 1, \ldots, 6$ and $a_k = 1/7$, $k = 1, \ldots, 7$, and $s \in \{0, 0.06\}$. To summarize the overall results, we can see from Tables II–V that for larger values of N the model with opinion leaders resembles the simulations rather precisely, regardless of the values for s and L. For any $t \geqslant 1$, we adopted the value

$$\sigma^2(t) \doteq \frac{\sum_{i=1}^{\text{runs}} \left[P_+^{\text{simul}(i)}(t) - \bar{P}_+(t) \right]^2}{\text{runs}} \qquad (12)$$

(where runs $= 500$) in order to summarize the variance of the results of our model (10), over the runs. In particular, $P_+^{\text{simul}(i)}(t)$ is the value $N_+(t)/N$ provided by the ith run of the simulation, and $\bar{P}_+(t)$ is exactly the mean value of $P_+(t)$ over the runs. Observe that for any $t \geqslant 1$ the values of $\sigma^2(t)$ are always quite small.

Figure 3 also reports the information in Tables II–V, with a different perspective. Here we summarized the results of the comparison between our model (10) and the simulation, by considering in particular the case $L = 7$, $s = 0.06$, and time step (t step) $= 9$, since it corresponds to the case of the *largest mismatch* between our proposal and the simulation. Observe that for larger values of N our proposal definitely improves the matching with the simulation. The lower picture in Fig. 3 also reports a similar comparison for the case $L = 6$, $s = 0.06$, t step $= 9$, and $N = 100, \ldots, 1000$. Again, the performance of our proposal improves when N increases. The latter results were expected, since a similar behavior was observed in Ref. [20] for the original Galam model (1).

We completed our simulations by also considering cases with large groups. More precisely we chose $N = 1000$ agents, setting $L = 20$, $a_k = 1/20$ for each $k = 1, \ldots, 20$, $N_+(0) = 510$, and $s = 25/1000 \approx 0.025$. It turned out that the latter case was not specifically favorable to our model. Nonetheless, for any $t \geqslant 1$ the value $\sigma^2(t)$ never exceeded 0.125×10^{-4}.

V. PROPERTIES OF THE OPINION LEADER MODEL

In this section we describe and prove specific results for the opinion leader model (10). To a large extent the reported properties encompass again some properties of model (1), as a special case. Using the same setting of the parameters adopted in Fig. 1, we begin by reporting in Fig. 4 (k is *odd* with

TABLE V. Comparison between the model (10) and a simulation, where we set respectively $L = 6$, $a_k = 1/6$, $k = 1, \ldots, 6$ or $L = 7$, $a_k = 1/7$, $k = 1, \ldots, 7$, and $s \in \{0, 0.06\}$ (similar results hold also for other values of s). The value $\sigma^2(t)$ represents, at time step t, the average square error between the model (10) and the simulation, over 500 runs. The simulation is performed setting $N = 100$; the results are reported for ten steps of the simulation. We set $N_+(0) = 45$ so that the initial percentage of $+$ agents [i.e., $P_+(0) = 45/100$] was below the killing point computed for $s = 0$. Finally, the results are averaged over 500 runs. The model provides satisfactory results, both for $L = 6$ and $L = 7$, even though for larger values of N (see Table II) the results were more favorable.

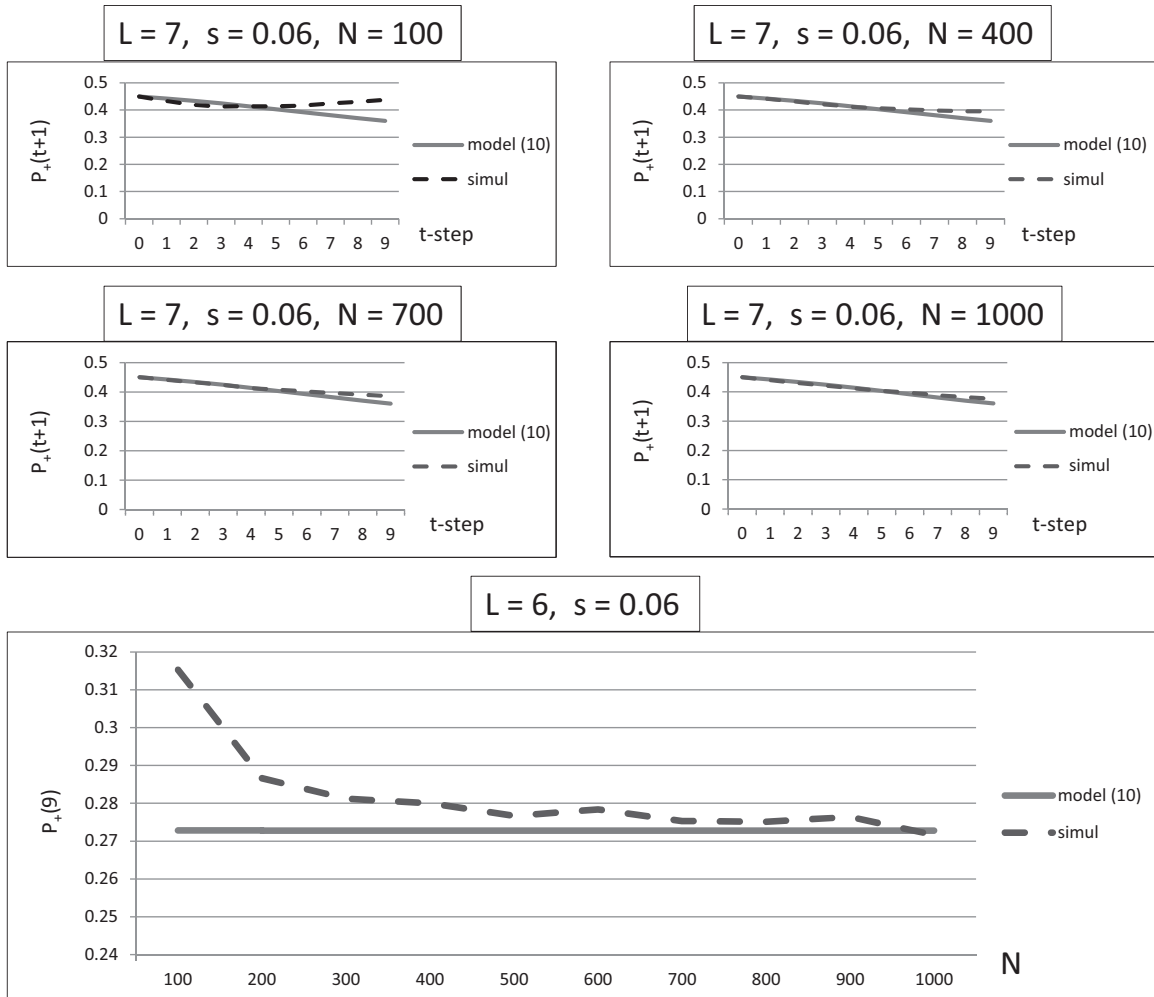|	$L = 6$, $s = 0$				$L = 6$, $s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3302	0.3273	$0.321 - 06$	1	0.4229	0.4189	$0.694 - 05$
2	0.1842	0.1840	$0.715 - 07$	2	0.3943	0.3931	$0.370 - 05$
3	0.0647	0.0720	$0.816 - 08$	3	0.3659	0.3721	$0.525 - 04$
4	0.0141	0.0206	$0.672 - 09$	4	0.3397	0.3502	$0.392 - 04$
5	0.0025	0.0048	$0.464 - 07$	5	0.3174	0.3426	$0.465 - 04$
6	0.0004	0.0007	$0.871 - 09$	6	0.3000	0.3258	$0.549 - 04$
7	0.0001	0.0001	$0.392 - 10$	7	0.2873	0.3197	$0.286 - 04$
8	0.0000	0.0000	$0.320 - 11$	8	0.2786	0.3183	$0.193 - 04$
9	0.0000	0.0000	$0.000 + 00$	9	0.2728	0.3153	$0.410 - 05$
	$L = 7$, $s = 0$				$L = 7$, $s = 0.06$		
t step	Model (10)	Simulation	$\sigma^2(t)$	t step	Model (10)	Simulation	$\sigma^2(t)$
0	0.4500	0.4500	$0.000 + 00$	0	0.4500	0.4500	$0.000 + 00$
1	0.3390	0.3355	$0.833 - 05$	1	0.4423	0.4333	$0.802 - 05$
2	0.1919	0.1907	$0.960 - 05$	2	0.4337	0.4184	$0.467 - 05$
3	0.0635	0.0731	$0.571 - 06$	3	0.4242	0.4128	$0.301 - 04$
4	0.0119	0.0189	$0.246 - 06$	4	0.4139	0.4135	$0.139 - 04$
5	0.0018	0.0042	$0.346 - 07$	5	0.4031	0.4138	$0.358 - 04$
6	0.0003	0.0009	$0.147 - 08$	6	0.3920	0.4164	$0.428 - 04$
7	0.0000	0.0001	$0.200 - 10$	7	0.3809	0.4244	$0.476 - 04$
8	0.0000	0.0000	$0.800 - 12$	8	0.3702	0.4302	$0.339 - 04$
9	0.0000	0.0000	$0.000 + 00$	9	0.3603	0.4372	$0.858 - 04$

FIG. 3. The top four panels refer to the data with $L = 7$ and $s = 0.06$ in Tables II–V, which correspond to the case of the largest mismatch between model (10) and the simulations (simul). We can observe that when N increases model (10) improves its matching with the simulation. The bottom panel represents the values of model (10) and the simulation at t step 9, for $L = 6$, $s = 0.06$, and $N = 100, \ldots, 1000$. Again we can observe that the performance of model (10) improves for larger values of N. A similar behavior was observed in Ref. [20] for the original Galam model (1).

$k \in \{5,7,9\}$) and Fig. 5 (k is *even* with $k \in \{4,6,8\}$), the graphs of $P_+(t + 1)$ vs $P_+(t)$ when $a_k = 1$ and $s \in \{0,0.02,0.04,0.05\}$ (i.e., when no opinion leader is included and when the probability of having opinion leaders is respectively raised to 2%, 4%, and 5%). Note that the graphs corresponding to $s = 0$ in Fig. 5 coincide with those in Fig. 1, and are reported only to allow an easy comparison with the extended model.

Observe that when $s \neq 0$ we do have to consider $P_+(t)$ values such that $P_+(t) \geqslant s$, since opinion leaders are a subset of the agents thinking $+$. This explains why the latter graphs are not defined for $P_+(t) < s$. Also note that in Fig. 5, where k is even, there is empirical evidence that when s increases the killing point progressively decreases, as one could expect.

Note that in the case $s = 0$ (i.e., using Galam's model), then $P_+(t + 1) = P_+(t)$ if $P_+(t) = 0$. On the contrary (see Figs. 4 and 5) in the case $s \neq 0$ then we can have a stationary point only if $P_+(t) > 0$ (which recalls some similar results in Ref. [22], where inflexible agents are considered).

Interestingly enough, the observation of Figs. 4(a), 4(b), 5(a), and 5(b) reveals also that there is a threshold value for s,

such that the graph of $P_+(t + 1)$ vs $P_+(t)$ becomes tangent to the line $P_+(t + 1) = P_+(t)$. When s is above the latter threshold the graph lies above the line $P_+(t + 1) = P_+(t)$ and the only (stable) fixed point is $\hat{P}_+ = 1$, regardless of the choice of the initial value $P_+(0)$. In particular, also observe that when s is below this threshold then, near the origin, the graph has a pair of fixed points (say, $P1$ and $P2$), corresponding to the intersection of the graph with the line $P_+(t + 1) = P_+(t)$. The one which is closer to the origin (say $P1$) is *stable*. Conversely, the fixed point $P2$, which is closer to the value $P_+(t) = 0.5$, is *unstable*. The latter observations are particularly remarkable, and could be fruitfully applied in practice.

Finally, the effect of opinion leaders is clearly more evident when k is large, i.e., when the discussions take place in large groups. In Fig. 6 we report a simulation of the latter effect, comparing the results when choosing, for example, the values $k = 5$ and $k = 21$.

Hereafter we are going to report specific properties of model (10), which play a key role. The proofs of the next propositions are detailed in the Appendix. First of all we can
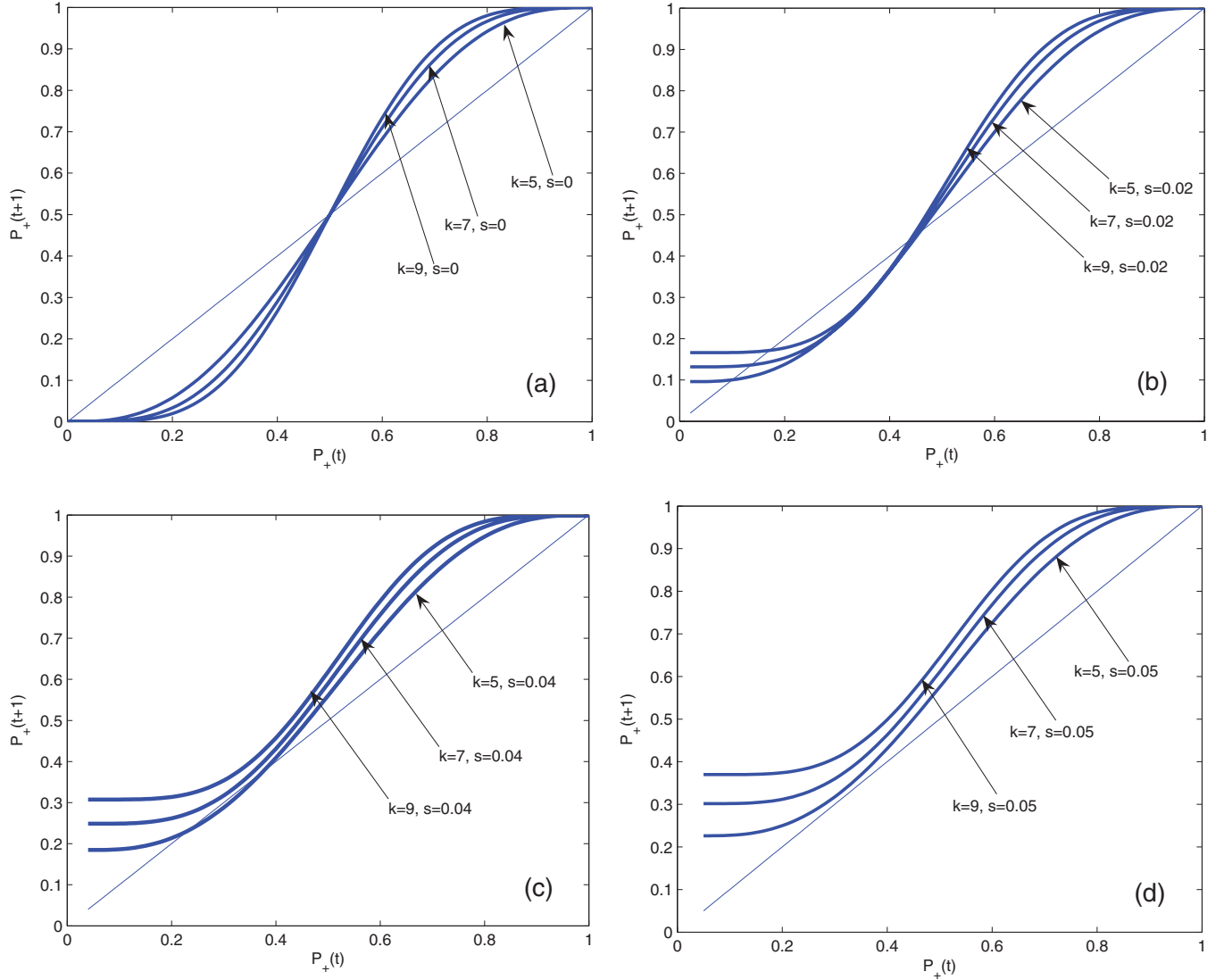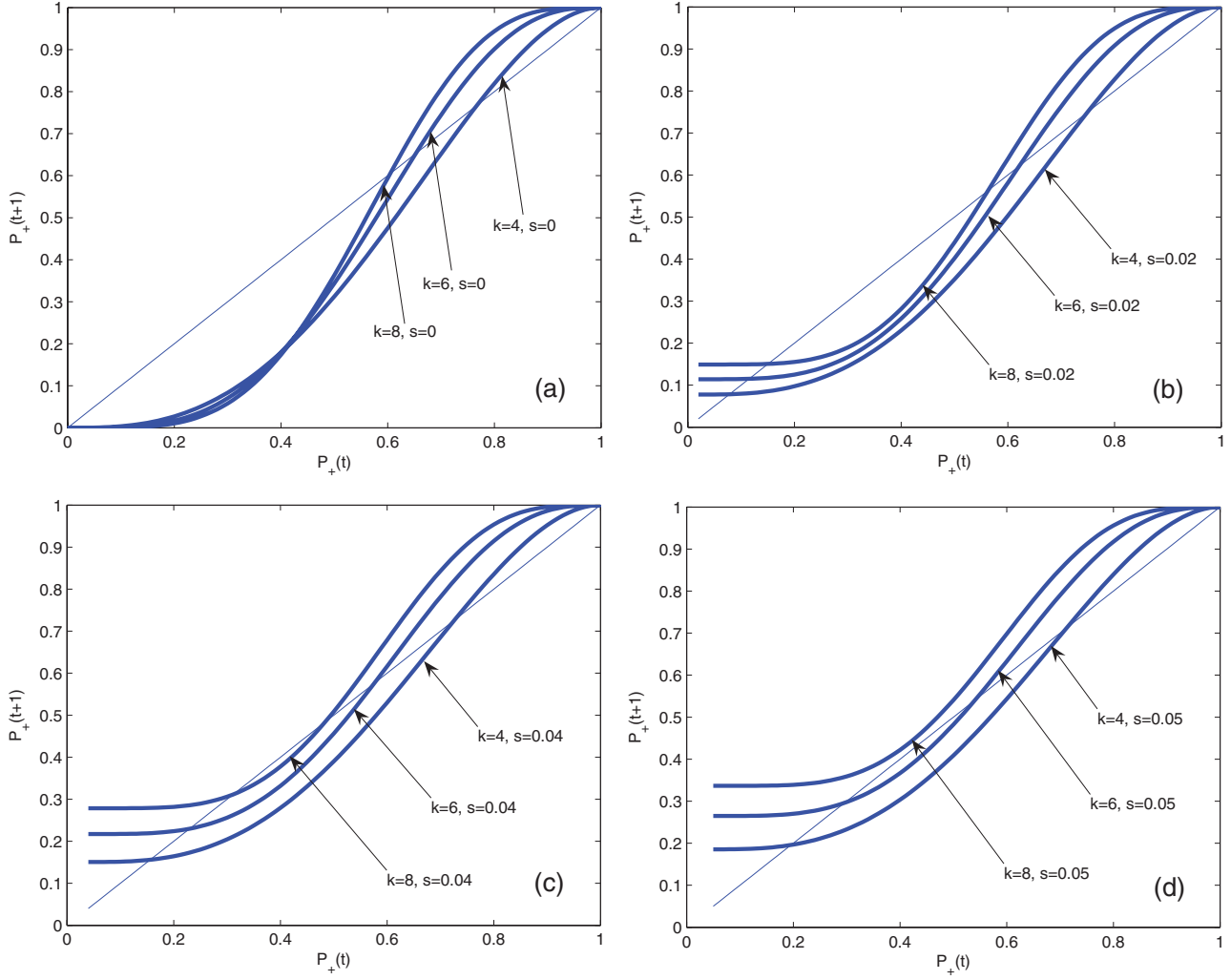
FIG. 4. (Color online) $P_+(t + 1)$ vs $P_+(t)$ as given by model (10), where the values of k and s are reported. We have $a_k = 1$, where k is odd, $k \in \{5,7,9\}$. The effect of opinion leaders is more evident when k is large.

prove, under mild assumptions, that the probability $P_+(t + 1)$ increases with $P_+(t)$.

Proposition 5.1. Given the model (10), if

$$0 \leqslant s \leqslant \min_{\substack{k = 2, \ldots, L \\ j = 1, \ldots, \lfloor \frac{k}{2} \rfloor}} \left\{ \frac{kP_+(t) - j}{k - j} \right\} \quad \text{for } P_+(t) \geqslant 0.5$$

(13)

then

$$\frac{\partial P_+(t + 1)}{\partial P_+(t)} \geqslant 0. \tag{14}$$

Moreover, if $a_1 \neq 0$ then (14) is satisfied as a strict inequality.

Observe that in Figs. 4 and 5 the curves have a horizontal tangent. The latter fact is confirmed by (A2) in the Appendix since, setting $a_1 = 0$, we have

$$\lim_{P_+(t) \to 1^-} \frac{\partial P_+(t + 1)}{\partial P_+(t)} = a_1 = 0.$$

Moreover, note that (14) concurs with the property *ii* of Corollary 3.2.

Proposition 5.2. Given the model (10), if $P_+(t) \neq 1$ we have

$$\frac{\partial P_+(t + 1)}{\partial s} > 0, \quad \forall \, t \geqslant 0.$$

Moreover, we have

$$\min_{s \geqslant 0} \left\{ \frac{\partial P_+(t + 1)}{\partial s} \right\} = \left. \frac{\partial P_+(t + 1)}{\partial s} \right|_{s = P_+(t)}$$

$$= \sum_{k=2}^{L} a_k C_1^k [1 - P_+(t)]^{k-1}, \quad (15)$$

$$\max_{s \geqslant 0} \left\{ \frac{\partial P_+(t + 1)}{\partial s} \right\} = \left. \frac{\partial P_+(t + 1)}{\partial s} \right|_{s = 0}.$$

The previous proposition substantially states that, as expected, the opinion leaders always tend to increase the probability to have agents who think $+$ at the subsequent time step.

FIG. 5. (Color online) $P_+(t + 1)$ vs $P_+(t)$ as given by model (10), where the values of k and s are reported. Similarly to Fig. 1 we have $a_k = 1$, where k is even, $k \in \{4,6,8\}$. The effect of opinion leaders is more evident when k is large.
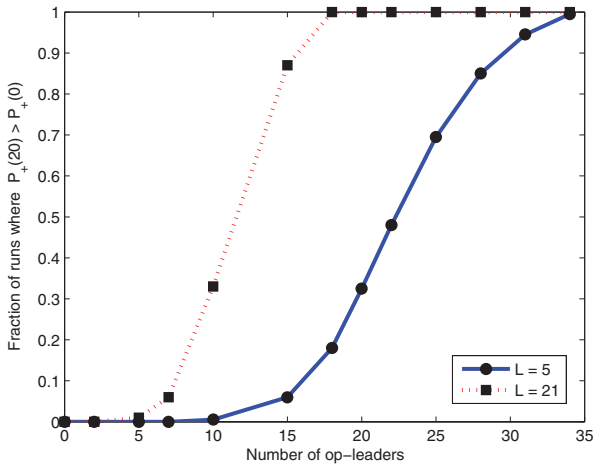

FIG. 6. (Color online) The effect of the number of opinion leaders (on the abscissa) on diffusion speedup, using the model (10). The two graphs represent the fraction of simulations (out of 200) where at step $t = 20$ we have $P_+(20) > P_+(0)$, estimating the fraction of runs converging towards 1. The graphs correspond to the case of $N = 1000$ agents, selecting in (10) $a_k = 1$, with $k = 21$ and $k = 5$, respectively.

Moreover, the least marginal effect corresponds to setting s at its maximum value, while the largest marginal effect is obtained on introducing the first opinion leader. Three simple considerations arise from Proposition 5.2:

(i) First, observe that when $a_L = 1$ and L increases, then there are intervals in which the quantity

$$\sum_{k=2}^{L} a_k C_1^k [1 - P_+(t)]^{k-1} = C_1^L [1 - P_+(t)]^{L-1}$$

in (15) decreases. In particular, it decreases when $L > 1/|\ln[1 - P_+(t)]|$, which is satisfied by any value of L when $P_+(t) \geqslant 0.5$. This means that when $a_L = 1$ and L is large, i.e., we allow larger groups of individuals, the marginal effect of opinion leaders is attenuated.

(ii) Second, since the minimum of $\partial P_+(t + 1)/\partial s$ is at $s = P_+(t)$ and the maximum is at $s = 0$, decreasing the value of s will increase the partial derivative $\partial P_+(t + 1)/\partial s$. Thus, a few opinion leaders are more "efficient" (although less effective) than many, in order to speed up the information exchange.
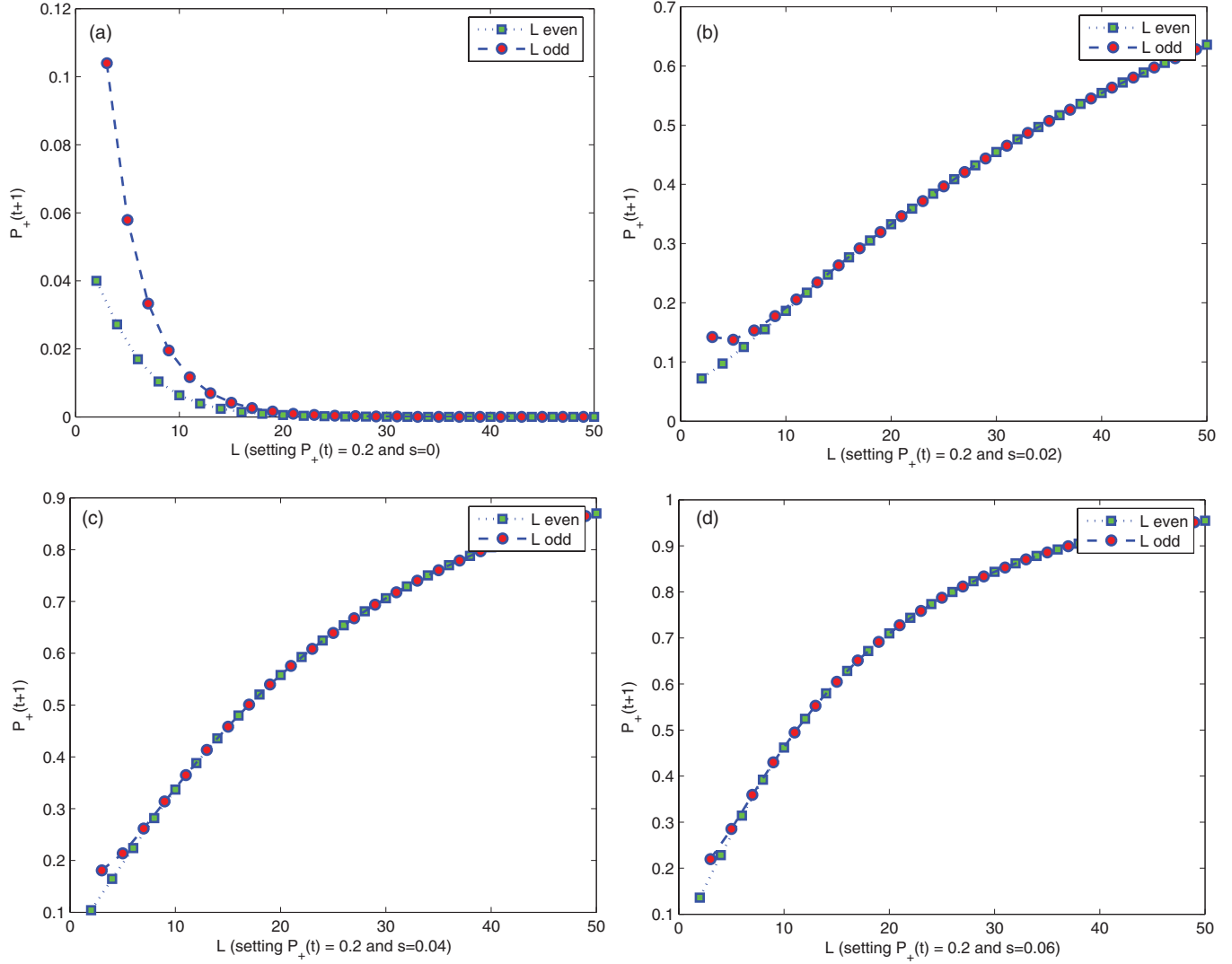
FIG. 7. (Color online) $P_+(t+1)$ vs L as given by model (10), where we set $a_L = 1$, $L \in \{2, \ldots, 50\}$, $P_+(t) = 0.2$, with $s \in \{0, 0.02, 0.04, 0.06\}$.

(iii) Third, increasing the value of s (starting from $s = 0$) will decrease the value of a possible killing point of (10). Indeed (see Figs. 4 and 5), since $P_+(t) - s$ is smaller than $P_+(t)$ and $\partial P_+(t+1)/\partial s > 0$, then increasing s will increase $P_+(t+1)$, so that the graphs in Figs. 4 and 5 move upwards. Note that since $P_+(t) - s < P_+(t)$, the same result can be obtained by simple inspection of (3) and (10).

Let us now address more explicitly the relation between the presence of opinion leaders, the maximum dimension L of a group, and $P_+(t)$ in (10). For this purpose, let us consider $P_+(t+1)$ as a function of s, L, and $P_+(t)$, where s is possibly a function of $P_+(t)$. We want to study the trade-off between the value of s with $P_+(t)$ and keeping $P_+(t+1)$ constant. In particular, by using the implicit function theorem we have

$$\frac{ds}{dP_+(t)} = -\frac{\frac{\partial P_+(t+1)}{\partial P_+(t)}}{\frac{\partial P_+(t+1)}{\partial s}}. \qquad (16)$$

Using (16) and Proposition 5.2 we conclude that

$$\left[\frac{ds}{dP_+(t)} \right] \left[\frac{\partial P_+(t+1)}{\partial P_+(t)} \right] < 0, \qquad (17)$$

which confirms the following reasoning: In order to leave $P_+(t+1)$ unchanged, if $\partial P_+(t+1)/\partial P_+(t)$ is positive [i.e., an increase of $P_+(t)$ determines an increase of the probability $P_+(t+1)$], a smaller number of opinion leaders will be necessary. The latter consideration yields a trade-off between the effects of the parameters s and $P_+(t)$ on $P_+(t+1)$.

A conclusion similar to (16) and (17) does not follow immediately when L is used in place of $P_+(t)$, since L is a *discrete parameter* and some care is mandatory when using generalizations of partial derivatives. Anyway, in order to possibly *estimate* the effects on $P_+(t+1)$ of increasing values for the parameter L [while $P_+(t)$ remains constant], in Figs. 7–10 we observe that the trends might be strongly dependent on the value of s. In particular, also observe from Figs. 7 and 8 that a slight increase of the percentage of opinion leaders reverses the final consensus of the entire group of
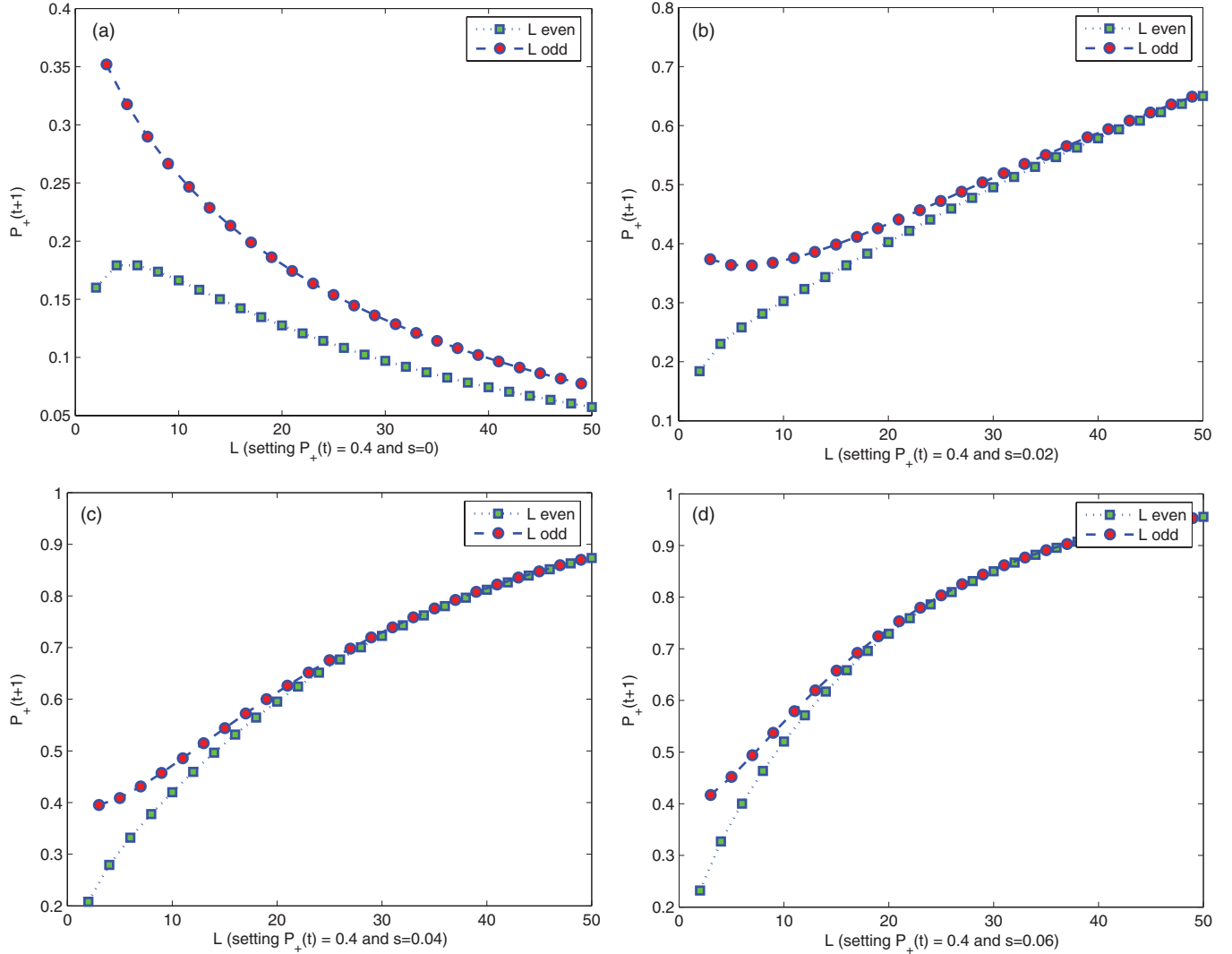
FIG. 8. (Color online) $P_+(t+1)$ vs L as given by model (10), where we set $a_L = 1$, $L \in \{2, \dots, 50\}$, $P_+(t) = 0.4$, with $s \in \{0, 0.02, 0.04, 0.06\}$.

agents. Moreover, convergence towards a positive consensus is strongly accelerated.

VI. CONCLUSIONS AND FUTURE WORK

This paper is first devoted to analyzing some properties of Galam's model [6], and then it introduces another model for agents' interactions. In particular, in this model special individuals, namely, opinion leaders, play a key role in information spreading when the majority rule defined in [6] holds.

We have described our model along with some remarkable properties, and a partial numerical investigation. In particular, the numerical tests highlight that the role of opinion leaders can strongly affect the dynamics of information spreading. We can summarize some relevant results for our proposal as follows:

(i) A few opinion leaders (say, 2%–5% of the N agents) may strongly accelerate the convergence towards the opinion + (see Figs. 4 and 5).

(ii) A few opinion leaders can possibly reverse the convergence process of the entire group of agents, so that convergence can be moved from opinion – to opinion + (see Fig. 8).

(iii) A few opinion leaders are more efficient (though less effective) than many, in order to speed up the information exchange (see Proposition 5.2).

(iv) Increasing the value of s (starting from $s = 0$) will decrease the value of a possible killing point of model (10) (see Figs. 4 and 5).

Possible generalizations of model (10) to the spreading of more than two opinions in the population could be studied (see also Ref. [23]).

Finally, we remark that the interest in agents' interactions and information spreading is motivated by their strategic impact on real problems. We will consider in a future work the application of (10), both to case studies from the literature and to real life situations, where opinion leaders may play an essential role. As an example, consider the possible application to wine purchasing reported in Ref. [19].
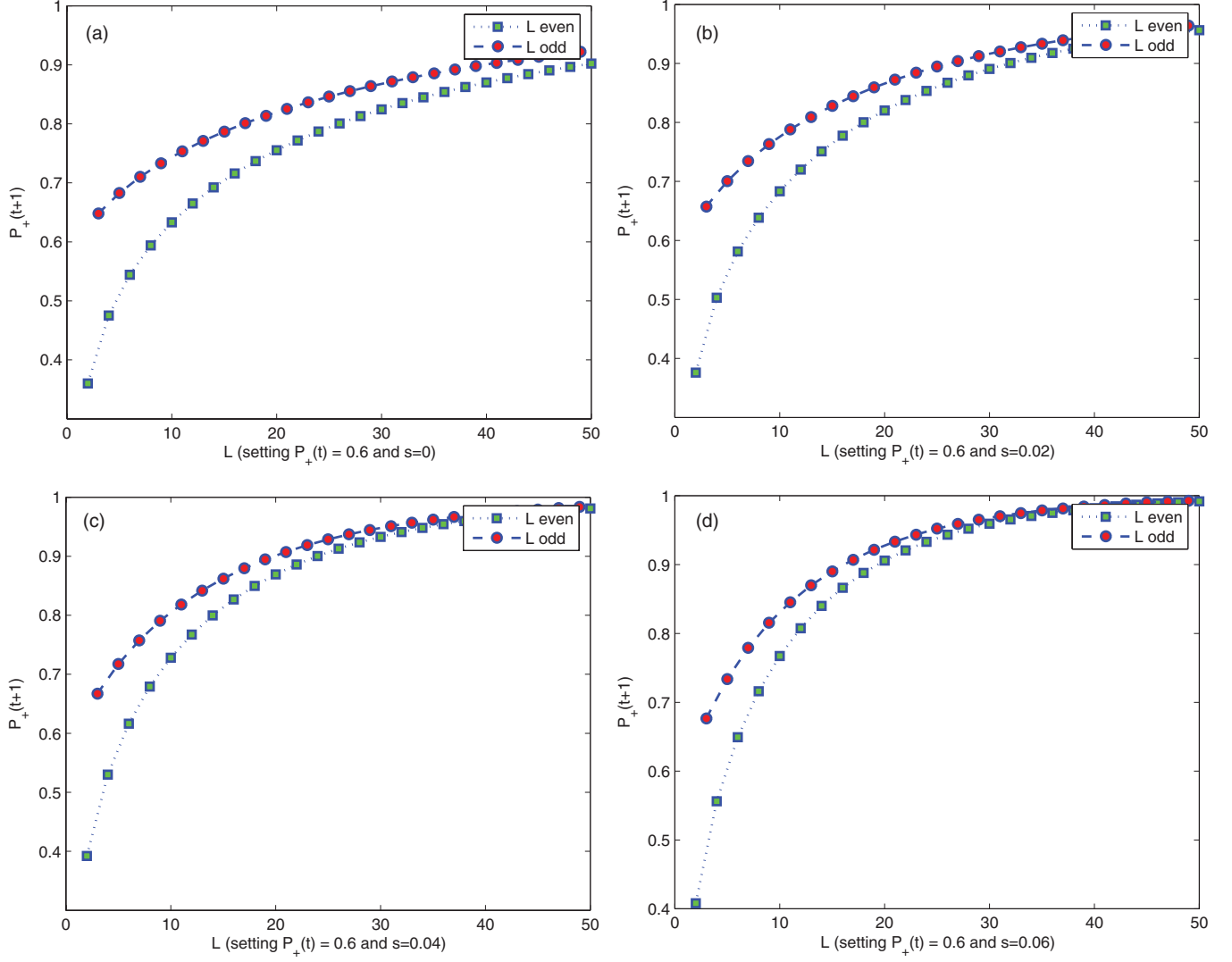
FIG. 9. (Color online) $P_+(t+1)$ vs L as given by model (10), where we set $a_L = 1$, $L \in \{2,\ldots,50\}$, $P_+(t) = 0.6$, with $s \in \{0,0.02,0.04,0.06\}$.

In addition, at least three generalizations of model (10) can be considered. Indeed, we may consider the following cases:

(i) When the number N of agents is small model (10) might be relatively inaccurate (see Fig. 3).

(ii) Besides opinion leaders whose opinion is $+$, opinion leaders thinking $-$ can also be introduced, in order to generalize the model (see also Galam's definition of inflexibles in Ref. [18]).

(iii) The idea of opinion leaders who convince all the agents joining the same group can be far too restrictive in some practical applications. Indeed, a more general model may be conceived where the opinion leaders convince the whole group, provided that the latter is not too large.

APPENDIX: (PROOFS OF RESULTS AND SOME SPECIAL CASES)

1. Proof of Proposition 3.1

Item 1 easily follows from Eq. (4). As regards 2, observe that $\mathcal{G}_k$ is a polynomial in $P_+(t)$, and by (8) and (4) we have

$$\frac{\partial \mathcal{G}_k[P_+(t)]}{\partial P_+(t)} = \frac{\partial}{\partial P_+(t)} \left[\frac{1}{\beta_k} \int_0^{P_+(t)} u^{\lfloor \frac{k}{2} \rfloor} (1-u)^{k-\lfloor \frac{k}{2} \rfloor - 1} du \right]$$

$$= \frac{1}{\beta_k} P_+(t)^{\lfloor \frac{k}{2} \rfloor} \{1 - P_+(t)\}^{k-\lfloor \frac{k}{2} \rfloor - 1}, \quad (A1)$$

which implies

$$\frac{\partial \mathcal{G}_k[P_+(t)]}{\partial P_+(t)} > 0 \quad \text{for any } P_+(t) \in (0,1).$$
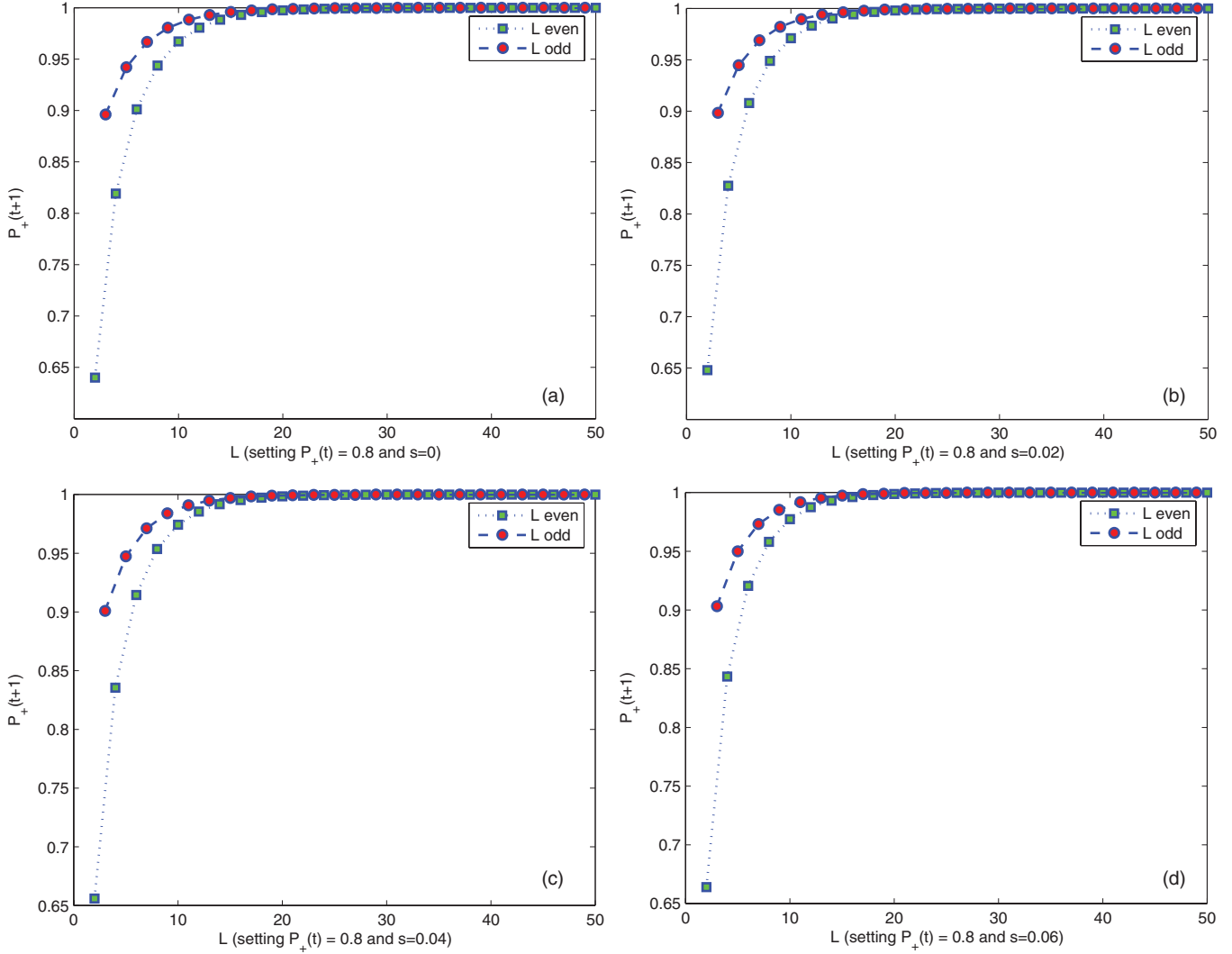
FIG. 10. (Color online) $P_+(t+1)$ vs L as given by model (10), where we set $a_L = 1$, $L \in \{2, \ldots, 50\}$, $P_+(t) = 0.8$, with $s \in \{0, 0.02, 0.04, 0.06\}$.

As regards 3, setting $k = 1$ in Eq. (4) we readily obtain the result. Now, for items (4) and (5) we split the proof, setting k either even or odd. We have for $k \geqslant 2$

$$
\frac{\partial^2 \mathcal{G}_k[P_+(t)]}{\partial P_+(t)^2}
$$
$$
= \frac{1}{\beta_k} \frac{\partial}{\partial P_+(t)} \left[P_+(t)^{\lfloor \frac{k}{2} \rfloor} \{1 - P_+(t)\}^{k - \lfloor \frac{k}{2} \rfloor - 1} \right]
$$
$$
= \frac{1}{\beta_k} \left[\left\lfloor \frac{k}{2} \right\rfloor P_+(t)^{\lfloor \frac{k}{2} \rfloor - 1} \{1 - P_+(t)\}^{k - \lfloor \frac{k}{2} \rfloor - 1} \right.
$$
$$
\left. - P_+(t)^{\lfloor \frac{k}{2} \rfloor} (k - \lfloor k/2 \rfloor - 1) \{1 - P_+(t)\}^{k - \lfloor \frac{k}{2} \rfloor - 2} \right]
$$

so that for k even

$$
\frac{\partial^2 \mathcal{G}_k[P_+(t)]}{\partial P_+(t)^2} = \frac{1}{\beta_k} P_+(t)^{\frac{k}{2} - 1} \{1 - P_+(t)\}^{\frac{k}{2} - 2}
$$
$$
\times \left\{ \frac{k}{2} [1 - P_+(t)] - \left(\frac{k}{2} - 1 \right) P_+(t) \right\}
$$

$$
= \frac{k!}{\left(\frac{k}{2} \right)! \left(\frac{k}{2} - 1 \right)!} P_+(t)^{\frac{k}{2} - 1} \{1 - P_+(t)\}^{\frac{k}{2} - 2}
$$
$$
\times \left[\frac{k}{2} + (1 - k) P_+(t) \right],
$$

which yields item (4). On the other hand, for $k \geqslant 3$ and k odd we obtain

$$
\frac{\partial^2 \mathcal{G}_k[P_+(t)]}{\partial P_+(t)^2} = \frac{1}{\beta_k} P_+(t)^{\frac{k-3}{2}} \{1 - P_+(t)\}^{\frac{k-3}{2}}
$$
$$
\times \left\{ \frac{k-1}{2} [1 - P_+(t)] - \left(\frac{k-1}{2} \right) P_+(t) \right\}
$$
$$
= \frac{k!}{\left(\frac{k-1}{2} \right)! \left(\frac{k-1}{2} \right)!} P_+(t)^{\frac{k-3}{2}} \{1 - P_+(t)\}^{\frac{k-3}{2}}
$$
$$
\times \left\{ \frac{k-1}{2} [1 - 2P_+(t)] \right\},
$$

which yields item (5). ∎

2. Proof of Corollary 3.1

If $k \neq 1$ the existence of a fixed point in $(0,1)$ follows from (A1). In fact, considering the function $h[P_+(t)] = \mathcal{G}_k[P_+(t)] - P_+(t)$ we have that $h(0) = h(1) = 0$; moreover

$$\lim_{P_+(t)\to 0^+} \frac{\partial h[P_+(t)]}{\partial P_+(t)} = \lim_{P_+(t)\to 0^+} \frac{\partial \mathcal{G}_k[P_+(t)]}{\partial P_+(t)} - 1 = -1,$$

and similarly $\lim_{P_+(t)\to 1^-} \partial h[P_+(t)]/\partial P_+(t) = -1$. Observing that, by Proposition 3.1, h is continuous on $[0,1]$, then h has at least one zero in $(0,1)$, which is unique due to items (4) and (5) of Proposition 3.1, i.e., $\mathcal{G}_k$ has a unique fixed point $\hat{P}_+$. The fixed point $\hat{P}_+$ is a killing point since, necessarily $h'(\hat{P}_+) > 0$, and thus $\mathcal{G}_k'(\hat{P}_+) > 1$, i.e., $\hat{P}_+$ turns out to be an unstable point while 0 and 1 are stable points of $\mathcal{G}_k$. ∎

3. Proof of Corollary 3.2

Properties (i), (ii), and (iii) follow immediately from items (1), (2), and (3) in Proposition 3.1 and the fact that $a_k \geqslant 0$, $k = 1, \ldots, L$. Items (iv), (v), and (vi) again follow from items (4) and (5) of Proposition 3.1 and the fact that the convex combination of convex functions is still convex. ∎

4. Proof of Corollary 3.3

If $a_1 = 1$ then $P_+(t+1) = P_+(t)$ for any $P_+(t) \in (0,1)$ due to item (iii) of Corollary 3.2. As in Corollary 3.1, if $a_1 < 1$ we define the function $H(P_+(t)) = \mathcal{G}(P_+(t)) - P_+(t)$ so that $H(0) = H(1) = 0$. Moreover,

$$\lim_{P_+(t)\to 0^+} \frac{\partial H[P_+(t)]}{\partial P_+(t)} = \lim_{P_+(t)\to 0^+} \frac{\partial \mathcal{G}[P_+(t)]}{\partial P_+(t)} - 1$$

$$= \sum_{k=1}^{L} a_k \lim_{P_+(t)\to 0^+} \frac{\partial \mathcal{G}_k[P_+(t)]}{\partial P_+(t)} - 1$$

$$= a_1 - 1 < 0;$$

similarly, we obtain $\lim_{P_+(t)\to 1^-} \partial H[P_+(t)]/\partial P_+(t) < 0$. Now, H is continuous on $[0,1]$ and therefore H has at least one zero in $(0,1)$, i.e., $\mathcal{G}$ has a fixed point in $(0,1)$. ∎

5. Proof of Corollary 3.4

For any k and any j, if $P_+(t) = 1/2$, then

$$\{P_+(t)\}^j \{1 - P_+(t)\}^{k-j} = \frac{1}{2^k} .$$

Moreover, for any k odd, the binomial theorem yields

$$\sum_{j=0}^{k} C_j^k = 2^k, \qquad \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} C_j^k = 2^{k-1},$$

and therefore $\mathcal{G}_k(1/2) = 1/2$. In this way, since $\sum_{k=1}^{L} a_k = 1$, also $\mathcal{G}(1/2) = 1/2$, i.e., $1/2$ is a fixed point for $\mathcal{G}$. Moreover, by Corollary 3.1 $\hat{P}_+ = 1/2$ is the killing point of $\mathcal{G}_k$, k odd; thus $1/2$ is also the killing point of $\mathcal{G}$. ∎

6. Proof of Proposition 5.1

Since (10) is continuously differentiable with respect to $P_+(t)$ we have for any s

$$\frac{\partial P_+(t+1)}{\partial P_+(t)} = -a_1 C_0^1 [1 - P_+(t)]^0 (-1) - \sum_{k=2}^{L} a_k \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} C_j^k$$

$$\times \{ j [P_+(t) - s]^{j-1} [1 - P_+(t)]^{k-j}$$

$$+ (k-j)[P_+(t) - s]^j [1 - P_+(t)]^{k-j-1} (-1) \}$$

$$- \sum_{k=2}^{L} a_k C_0^k k [1 - P_+(t)]^{k-1} (-1),$$

or equivalently

$$\frac{\partial P_+(t+1)}{\partial P_+(t)} = \sum_{k=1}^{L} a_k C_0^k k [1 - P_+(t)]^{k-1} + \sum_{k=2}^{L} a_k \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} C_j^k$$

$$\times \{ -j [P_+(t) - s]^{j-1} [1 - P_+(t)]^{k-j}$$

$$+ (k-j)[P_+(t) - s]^j [1 - P_+(t)]^{k-j-1} \},$$

$$(A2)$$

which is the sum of all non-negative terms as long as

$$(k-j)[P_+(t) - s] \geqslant j[1 - P_+(t)],$$

$$k = 2, \ldots, L, \quad j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor,$$

i.e.,

$$s \leqslant \frac{kP_+(t) - j}{k - j}, \quad k = 2, \ldots, L, \quad j = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor,$$

which yields the conditions (13).

Finally, if $a_1 \neq 0$ and (13) holds then (A2) satisfies (14) as a strict inequality also when $P_+(t) = s$ or $P_+(t) = 1$. ∎

7. Proof of Proposition 5.2

Observe that (10) is continuously differentiable with respect to s and after a trivial computation we obtain for any $P_+(t) > s$

$$\frac{\partial P_+(t+1)}{\partial s} = -\sum_{k=2}^{L} a_k \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} j C_j^k$$

$$\times [P_+(t) - s]^{j-1} (-1)[1 - P_+(t)]^{k-j},$$

which is the sum of all non-negative terms and at least one strictly positive term (corresponding to $j = 1$). Furthermore, by simple inspection, $\partial P_+(t+1)/\partial s$ decreases when s increases in the range $[0, P_+(t)]$, and thus relations (15) hold. ∎

Hereafter, in this section we compute the killing point of (1), for the special cases in which the largest dimension of the groups ranges from $L = 1$ to $L = 4$. The latter cases are very frequent in practical applications and arise in a variety of real situations (see, e.g., Refs. [18,19]).

For this purpose, let us consider the following expression from the binomial theorem, where $x \in \mathbb{R}$ and $k \geqslant j$, with k, j

integers:

$$
(1-x)^{k-j} = \sum_{h=0}^{k-j} \binom{k-j}{h} 1^h (-x)^{k-j-h}
$$

$$
= \sum_{h=0}^{k-j} \binom{k-j}{h} (-1)^{k-j-h} x^{k-j-h}.
$$

Hence, from (1) when $0 \leqslant P_+(t) \leqslant 1$ we have

$$
\mathcal{G}(P_+(t)) = \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j}
$$

$$
= \sum_{k=1}^{L} \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} \sum_{h=0}^{k-j} (-1)^{k-j-h} a_k
$$

$$
\times \binom{k}{j} \binom{k-j}{h} P_+(t)^{k-h}. \qquad \text{(A3)}
$$

Lemma A.1. Given the positive integer L and the relation $\sum_{k=1}^{L} a_k = 1$, with $a_k \geqslant 0$, $k = 1, \ldots, L$, for any $L \geqslant 1$ the function $\mathcal{G}$ in (A3) has at least the two fixed points $P_+(t) = 0$ and $P_+(t) = 1$.

Proof. Equation (A3) is homogeneous with respect to $P_+(t)$ so that $P_+(t) = 0$ is clearly a solution. In addition, from (2) and since

$$
\lim_{P_+(t) \to 1^-} \{1 - P_+(t)\}^{k-j} = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j, \end{cases}
$$

we have

$$
\lim_{P_+(t) \to 1^-} \sum_{k=1}^{L} a_k \sum_{j=\lfloor \frac{k}{2}+1 \rfloor}^{k} C_j^k P_+(t)^j \{1 - P_+(t)\}^{k-j} = P_+(t),
$$

which proves that $P_+(t) = 1$ is again another fixed point of $\mathcal{G}$. ∎

Now, setting $L \in \{1,2,3,4\}$ in (A3) we want to compute the killing point of function $\mathcal{G}$. The cases where $L = 1, \ldots, 4$ are important since practical problems, where small groups of individuals are involved, are quite common.

Let us examine the trivial case $L = 1$. This implies that the subsets of people have just one member; thus, each of the individuals will preserve his or her initial opinion. The killing point corresponds to $\hat{P}_+ = 1$.

Let now $L = 2$ (the *dance hall* problem). According to (A3) with $L = 2$ we have

$$
\mathcal{G}(P_+(t))
$$

$$
= \sum_{j=\lfloor \frac{1}{2}+1 \rfloor}^{1} \sum_{h=0}^{1-j} (-1)^{1-j-h} a_1 \binom{1}{j} \binom{1-j}{h} P_+(t)^{1-h}
$$

$$
+ \sum_{j=\lfloor \frac{2}{2}+1 \rfloor}^{2} \sum_{h=0}^{2-j} (-1)^{2-j-h} a_2 \binom{2}{j} \binom{2-j}{h} P_+(t)^{2-h}
$$

$$
= a_1 P_+(t) + a_2 P_+(t)^2.
$$

The killing point may be determined using relation $\sum_{i=0}^{L} a_k = 1$ and by solving the equation

$$
P_+(t) = a_1 P_+(t) + a_2 P_+(t)^2.
$$

If $a_2 = 0$ we fall into the previous case where $L = 1$. Otherwise, we have only the two stationary points 0 and 1 (see Lemma A.1), where only 1 is the killing point $\hat{P}_+$.

When $L = 3$ relation (A3) becomes

$$
\mathcal{G}(P_+(t))
$$

$$
= \sum_{j=\lfloor \frac{1}{2}+1 \rfloor}^{1} \sum_{h=0}^{1-j} (-1)^{1-j-h} a_1 \binom{1}{j} \binom{1-j}{h} P_+(t)^{1-h}
$$

$$
+ \sum_{j=\lfloor \frac{2}{2}+1 \rfloor}^{2} \sum_{h=0}^{2-j} (-1)^{2-j-h} a_2 \binom{2}{j} \binom{2-j}{h} P_+(t)^{2-h}
$$

$$
+ \sum_{j=\lfloor \frac{3}{2}+1 \rfloor}^{3} \sum_{h=0}^{3-j} (-1)^{3-j-h} a_3 \binom{3}{j} \binom{3-j}{h} P_+(t)^{3-h}
$$

$$
= a_1 P_+(t) + a_2 P_+(t)^2 - 3a_3 P_+(t)^3
$$

$$
+ 3a_3 P_+(t)^2 + a_3 P_+(t)^3
$$

$$
= a_1 P_+(t) + (a_2 + 3a_3) P_+(t)^2 - 2a_3 P_+(t)^3.
$$

As Lemma A.1 stated, we see that $P_+(t) = 0$ is a fixed point of $\mathcal{G}$. Furthermore, if $a_3 \neq 0$ the other two fixed points are given by

$$
\frac{1}{4a_3} \left[a_2 + 3a_3 + \left(a_3^2 - 2a_2 a_3 + a_2^2 \right)^{1/2} \right]
$$

$$
= \frac{1}{4a_3} \left[a_2 + 3a_3 + (a_3 - a_2) \right] = 1,
$$

$$
\frac{1}{4a_3} \left[a_2 + 3a_3 - \left(a_3^2 - 2a_2 a_3 + a_2^2 \right)^{1/2} \right]
$$

$$
= \frac{1}{4a_3} \left[a_2 + 3a_3 - (a_3 - a_2) \right] = \frac{a_2 + a_3}{2a_3} \geqslant \frac{1}{2}.
$$

The latter formulas confirm two obvious considerations. First, they match the statement of Lemma A.1. Then, if the *majority rule* is adopted within each subset of individuals (with a bias for − in case of ties), then the killing point $\hat{P}_+ = (a_2 + a_3)/(2a_3)$ is larger than 0.5. Note also that in the case $a_2 = 0$ the results of Corollary 3.4 trivially hold.

When $L = 4$ relation (A3) becomes

$$
\mathcal{G}(P_+(t))
$$

$$
= \sum_{j=\lfloor \frac{1}{2}+1 \rfloor}^{1} \sum_{h=0}^{1-j} (-1)^{1-j-h} a_1 \binom{1}{j} \binom{1-j}{h} P_+(t)^{1-h}
$$

$$
+ \sum_{j=\lfloor \frac{2}{2}+1 \rfloor}^{2} \sum_{h=0}^{2-j} (-1)^{2-j-h} a_2 \binom{2}{j} \binom{2-j}{h} P_+(t)^{2-h}
$$

$$+ \sum_{j=\lfloor \frac{3}{2}+1 \rfloor}^{3} \sum_{h=0}^{3-j} (-1)^{3-j-h} a_3 \binom{3}{j} \binom{3-j}{h} P_+(t)^{3-h}$$

$$+ \sum_{j=\lfloor \frac{4}{2}+1 \rfloor}^{4} \sum_{h=0}^{4-j} (-1)^{4-j-h} a_4 \binom{4}{j} \binom{4-j}{h} P_+(t)^{4-h}$$

$$= a_1 P_+(t) + (a_2 + 3a_3) P_+(t)^2 + (4a_4 - 2a_3) P_+(t)^3$$
$$- 3a_4 P_+(t)^4.$$

In order to compute possible killing points for the case $L = 4$ we consider the solution of the equation

$$P_+(t) = a_1 P_+(t) + (a_2 + 3a_3) P_+(t)^2$$
$$+ (4a_4 - 2a_3) P_+(t)^3 - 3a_4 P_+(t)^4$$

or equivalently

$$(a_1 - 1) P_+(t) + (a_2 + 3a_3) P_+(t)^2 + (4a_4 - 2a_3) P_+(t)^3$$
$$- 3a_4 P_+(t)^4 = 0.$$

Again, as stated by Lemma A.1, we see that $P_+(t) = 0$ is a solution. Furthermore, since $P_+(t) = 1$ must also be

a fixed point of $\mathcal{G}$, by a simple polynomial division we obtain

$$(a_1 - 1) P_+(t) + (a_2 + 3a_3) P_+(t)^2$$
$$+ (4a_4 - 2a_3) P_+(t)^3 - 3a_4 P_+(t)^4$$
$$= P_+(t)[P_+(t) - 1][3a_4 P_+(t)^2$$
$$+ (2a_3 - a_4) P_+(t) + (a_1 - 1)],$$

whose zeros (i.e., possible killing points of $\mathcal{G}$) are

$$\frac{1}{6a_4} \{ a_4 - 2a_3 + [(2a_4 + 2a_3)^2 + 3a_4(3a_4 + 4a_2)]^{1/2} \}$$
$$> \frac{4a_4}{6a_4} = \frac{2}{3}, \tag{A4}$$
$$\frac{1}{6a_4} \{ a_4 - 2a_3 - [(2a_4 + 2a_3)^2 + 3a_4(3a_4 + 4a_2)]^{1/2} \} < 0.$$

Since the left-hand side of the last inequality is negative it cannot be a killing point. Finally, also observe that in the case $a_4 = 1$ (i.e., only groups with four individuals are allowed), then the killing point is $\hat{P}_+ = (1 + \sqrt{13})/6 \approx 0.768$ (as proven also in Ref. [19]).

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