Particle Swarm Optimization: dynamic system analysis for parameter selection in global optimization frameworks

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Abstract

In this paper we consider the evolutionary Particle Swarm Optimization (PSO) algorithm, for the minimization of a computationally costly nonlinear function, in global optimization frameworks. We study a reformulation of the standard iteration of PSO [KE95, CK02] into a linear dynamic system. Then, the latter is partially investigated in order to provide indications for the parameters selection in PSO. We carry out our analysis on a generalized PSO iteration, which includes the standard one proposed in the literature. In our scheme the path of any particle is possibly affected by the trajectories of all the other particles in the swarm. Then, for any particle we give indications on both the starting point and the trajectory, in terms of the PSO coefficients. We analyze the scheme assuming either deterministic and uniformly randomly distributed coefficients. Convergence analysis is partially provided.

Keywords : Global optimization, Evolutionary optimization, Particle Swarm Optimization, Dynamic linear system, Convergence analysis.

1 Introduction

This paper is concerned with the solution of the global optimization problem

$$\min_{x\in\mathcal{L}} f(x), \qquad \mathcal{L}\subset \mathbb{R}^n, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a *continuous function*, and $\mathcal{L}$ is a *robust* and *bounded* set. In particular, we aim at detecting a global minimum x^* of (1.1), such that $f(x^*) \leq f(x)$ for any $x \in \mathcal{L}$, and the function is computationally costly. Examples of inherently difficult problems like (1.1) arise in several real aplications, e.g. in the aerospace, automotive and naval engineering, where Navier-Stokes solvers for shape design are used. Here, the CPU time for one function evaluation may require from one to ten hours, depending on the geometry details [PPC04].
As well known, unlike local optimization frameworks, there are not optimality conditions which can characterize the point x^*. Indeed, no stopping criterion both theoretical and practical, may be adopted in order to detect the global minima of problem (1.1). Anyway, several approaches have been proposed in the literature for the solution of (1.1) (see [P96] and therein references), either adopting deterministic [HT90] and/or stochastic [HPT00] techniques. Deterministic methods selectively exploit the local information on the objective function and aim at extending their exploration over a wide and possibly *dense* subset of $\mathcal{L}$.
Similarly, stochastic techniques randomly generate a sequence $\{x_k\}$ of points uniformly distributed in $\mathcal{L}$.

Then, under reasonable assumptions they provide asymptotic convergence results in terms of probability. A typical convergence result involves a subsequence of points $\{x_k\}_{\mathcal{K}}$ according with (see [P96])

$$Pr\left\{\lim_{k\to\infty,\ k\in\mathcal{K}} \rho(x_k, X^*) = 0\right\} = 1,$$

where X^* is the set of global minima of $f(x)$ in $\mathcal{L}$, $\rho(\cdot,\cdot)$ is a *distance* and $Pr\{\cdot\}$ indicates the *probability*. Evidently, the use of the latter techniques requires the exploration over a possibly dense subset of $\mathcal{L}$; thus, the increasing computational burden would be definitely unaffordable for the shape design problems described above.

In the last decades both deterministic and stochastic methods for problem (1.1) have included many contributions, whose rationale is often suggested by biological systems and/or social behavioural interpretations. In particular, the latter methods often disregard convergence analysis and focus on the heuristic solution of the problem in hand. Examples of such iterative approaches are *genetic algorithms*, where a set of points (namely the *population*) is chosen in the feasible set. Then, to update the population at current step, cyclic simple operations as *crossover (recombination)*, *selection* and *mutation* are performed on the points [T].

In a similar fashion the *evolutionary algorithms* consider a population of *individuals* (points) in the feasible set. Then, the population is updated by modifying each individual, according with the information on the objective function collected by the entire population. In this paper we analyze the PSO algorithm [KE95], an iterative method for global optimization, in the class of evolutionary algorithms. The latter scheme was originally conceived in '95 as a model to describe the behaviour of a flock of birds. Then, it has been recently improved (see also [CK02, OM99, SE98, ZMZQ03, CLLPP05]) and is widely adopted within specific global optimization applications [BWGHM00, FG00, PPC04, VS04].
The growing interest for PSO algorithm, to solve distinctive global optimization problems (e.g. ship design), is encouraged by the following appealing features:

- *balance* between the computation involved and the precision of the solution detected;
- constant *computational cost* and *memory engagement* at each iteration;
- availability of a *current approximate solution*;
- *derivatives* of the objective function not required;
- easy *implementation* and *parallelization* of the method.

We recall that PSO iteration is neither able to guarantee the convergence to a global minimum nor to a local minimum. Indeed, PSO is a heuristic method, and its reformulations in the literature are heuristics as well. The general PSO iteration can be described at step $k \geq 0$ by

$$x_j^{k+1} = x_j^k + d_j^k, \qquad j = 1, \ldots, P, \tag{1.2}$$

where $x_j^k \in \mathcal{L}$ is the current position of the j-th individual (*particle*) of the population (*swarm*), $d_j^k \in \mathbb{R}^n$ is the search direction and x_j^{k+1} is the new position at step $k+1$. Observe that d_j^k may *not* be a descent direction for the objective function $f(x)$ at x_j^k. In particular, direction d_j^k depends on both the search direction d_j^{k-1} and the paths of the particles. The original version of PSO in [KE95] considers at step k the direction

$$d_j^k = d_j^{k-1} + \alpha^k(x_j^{\text{best}} - x_j^k) + \beta^k(x^{\text{best}} - x_j^k), \tag{1.3}$$

where α^k, β^k are suitable random scalars and

$$x_j^{\text{best}} = \operatorname*{argmin}_{0\leq h\leq k} \{f(x_j^h)\}, \qquad j = 1, \ldots, P$$

$$x^{\text{best}} = \operatorname*{arg\,min}_{0\leq h\leq k,\ j=1,\ldots,P} \{f(x_j^h)\}.$$

In particular, d_j^{k-1} contributes to maintain the search direction of step $k-1$ while α^k and β^k (respectively called *cognitive* and *social* parameter) have different roles. With α^k the contribution from the history of the

j-th particle, summarized in the vector $x_j^{\text{best}} - x_j^k$, is suitably considered. On the other hand, β^k weights the influence of all the particles in the swarm (the vector $x^{\text{best}} - x_j^k$) on the j-th particle.

In this paper we figure out a partial analysis of the particles trajectories, on a very general reformulation of PSO iteration. Our approach is not completely new in the literature (similar issues are partially considered and investigated in [MS04, CK02, KE95, OM99]). First we extend several results from the literature; then, we provide our analysis within a framework of preliminary convergence. In particular, we reformulate our PSO generalized iteration into a dynamic linear system, where feedback is not included. The lack of a feedback block is a simplification, motivated by a couple of basic reasons. On one hand, though preferable, a complete stability analysis of a closed-loop system would be far from our purposes. Furthermore, some practical results may be obtained by only focusing on the free response of the open-loop system.

We defer to a second paper a complete numerical experience, on our generalized version of PSO algorithm. The latter paper will also contain a numerical comparison with the current literature.

In this paper $\mathbf{Z}$ and $\mathbf{C}$ respectively represent the set of integers and complex numbers. With '$\text{rk}(A)$' we indicate the rank of matrix A and '$\det(A)$' is its determinant. Finally we denote with I the identity matrix of suitable dimension.

In Section 2 we describe our generalized PSO iteration by means of a dynamic linear system, whose properties are partially analyzed in Sections 3 and 3.1. Section 4 is devoted to investigate promising starting points for the particles, while in Section 5 we study a generalization of our model. Section 6 considers some theoretical issues about our proposal and Section 7 provides some partial convergence results. In particular, the Section 7 addresses the PSO parameters selection by imposing that the particles trajectories are confined in a suitable compact set. Finally, a section of conclusions completes the paper.

2 A dynamic model for PSO

Consider the following (generalized) iteration of PSO algorithm in (1.2)-(1.3), which is reported by using a common notation adopted in the literature:

$$\begin{cases} v_j^{k+1} = \chi \left[w^k v_j^k + c_j r_j (p_j^k - x_j^k) + c_g r_g (p_g^k - x_j^k) \right], & k = 0, 1, ... \\ x_j^{k+1} = x_j^k + v_j^{k+1}, & k = 0, 1, ..., \end{cases} \tag{2.1}$$

where $j = 1, ..., P$ indicates the j-th particle, P is finite, v_j^k and x_j^k are the *speed* and the *position* of particle j at step k, p_j^k and p_g^k respectively satisfy

$$p_j^k = \text{argmin}_{l \le k} \left\{ f(x_j^l) \right\}, \qquad p_g^k = \text{argmin}_{l \le k,\ j=1,...P} \left\{ f(x_j^l) \right\}, \tag{2.2}$$

and $\chi, w^k, c_j, r_j, c_g, r_g$ are suitable bounded coefficients. *We assume at present that the latter parameters are real constants* with $0 \le r_j \le 1$ and $0 \le r_g \le 1$. The latter position will be generalized in Section 8 with the introduction of uniformly distributed random parameters.
Then, we can further generalize (2.1) by assuming that possibly the speed v_j^{k+1} depends on all the terms $p_h^k - x_j^k$, $h = 1, \ldots, P$, and not only on vectors $p_j^k - x_j^k$, $p_g^k - x_j^k$. In words this corresponds to allow a more general *social* contribution in the PSO iteration. The new iteration is therefore

$$\begin{cases} v_j^{k+1} = \chi_j^k \left[w_j^k v_j^k + \sum_{h=1}^{P} c_{h,j}^k r_{h,j}^k (p_h^k - x_j^k) \right], & k = 0, 1, ... \\ x_j^{k+1} = x_j^k + v_j^{k+1} & k = 0, 1, ... , \end{cases} \tag{2.3}$$

where $c_{h,j}^k$ and $r_{h,j}^k$ depend on the step (k), the current particle (j) and the other particles (h).
Without loss of generality at present we focus on the j-th particle and omit the subscript in the recurrence (2.3). Moreover, at present time we assume in (2.3) $\chi_j^k = \chi$, $c_{h,j}^k = c_h$, $r_{h,j}^k = r_h$ and $w_j^k = w$, for any $k \ge 0$. This is a common hypothesis in the PSO literature, which will be removed in Sections 4, 5 and 7, where we

will extend our results including the case of non-constant parameters, for any k. With the latter position the iteration (2.3) is equivalent to the *dynamic discrete, linear* and *stationary (time-invariant) system*

$$X(k+1) = \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix} X(k) + \begin{pmatrix} \sum_{h=1}^{P} \chi c_h r_h p_h^k \\ \\ \sum_{h=1}^{P} \chi c_h r_h p_h^k \end{pmatrix}, \tag{2.4}$$

where

$$X(k) = \begin{pmatrix} v^k \\ \\ x^k \end{pmatrix} \in \mathbb{R}^{2n}, \qquad k \geq 0.$$

The authors are aware that (2.4) may be inappropriate for a full analysis of stability, since the vector p_h^k is substantially a feedback term of the system (indeed p_h^k depends on the sequences $\{x_h^k\}$). However, as described in the sequel, modelling the feedback of system (2.4) (i.e. modelling the expression of p_h^k) is more or less equivalent to resort to a deep knowledge of the objective function $f(x)$. Unfortunately, the latter is only known at points x_h^k. Nonetheless, instead of studying the ambitious issue of the stability, for a closed-loop model of system (2.4), we aim at considering the more tractable free response of (2.4) (see also [CK02]). Then, the latter is used to obtain some partial indications on the assessment of PSO coefficients. We give evidence in the sequel that this approach, though not complete, is effective in the applications.

The sequence $\{X(k)\}$ identifies a trajectory in the state space $\mathbb{R}^{2n}$, and since (2.4) is a linear and stationary system, we may consider the *free response* $X_{\mathcal{L}}(k)$ and the *forced response* $X_{\mathcal{F}}(k)$ of the trajectory $\{X(k)\}$. Then, considering (2.4) we explicitly obtain at step $k \geq 0$ [S97]

$$X(k) = X_{\mathcal{L}}(k) + X_{\mathcal{F}}(k), \tag{2.5}$$

where

$$X_{\mathcal{L}}(k) = \Phi(k) X(0), \qquad X_{\mathcal{F}}(k) \ = \ \sum_{\tau=0}^{k-1} H(k-\tau) U(\tau), \tag{2.6}$$

and (after few calculation)

$$\Phi(k) \ = \ \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix}^k \in \mathbb{R}^{2n \times 2n}, \tag{2.7}$$

$$H(k-\tau) \ = \ \begin{pmatrix} \chi w I & -\sum_{h=1}^{P} \chi c_h r_h I \\ \\ \chi w I & \left(1 - \sum_{h=1}^{P} \chi c_h r_h\right) I \end{pmatrix}^{k-\tau-1} \in \mathbb{R}^{2n \times 2n}, \tag{2.8}$$

$$U(\tau) \ = \ \begin{pmatrix} \sum_{h=1}^{P} \chi c_h r_h p_h^\tau \\ \\ \sum_{h=1}^{P} \chi c_h r_h p_h^\tau \end{pmatrix} \in \mathbb{R}^{2n}. \tag{2.9}$$

Observe that $X_{\mathcal{L}}(k)$ in (2.6) does not depend on the vector p_h^k, but uniquely on the initial point $X(0)$. On the contrary, $X_{\mathcal{F}}(k)$ in (2.6) depends on the vector p_h^k and is independent of $X(0)$. The latter observation allows us to compute separately the two terms and will be heavily used later on. In order to carry out our conclusions, in the next two sections we compute $X_{\mathcal{L}}(k)$ by investigating the eigenpairs of matrix $\Phi(k)$ in (2.7).

3 Computation of the free response $X_{\mathcal{L}}(k)$

Let us consider the following position in (2.7) and (2.8)

$$a = \chi w, \qquad \omega = \sum_{h=1}^{P} \chi c_h r_h, \tag{3.1}$$

so that for any $\lambda \in \mathbf{C}$

$$\Phi(1) - \lambda I = \begin{pmatrix} (a-\lambda)I & -\omega I \\ aI & (1-\omega-\lambda)\, I \end{pmatrix},$$

whose Schur complement is given by $(1-\omega-\lambda+a\omega/(a-\lambda))I$. We aim at computing the $2n$ eigenvalues $\lambda^{(1)}, \ldots, \lambda^{(2n)}$ of $\Phi(1)$; thus, assuming for a while $\lambda^{(l)} \neq a$, $l = 1, \ldots, 2n$, we obtain

$$\begin{aligned} \det\left(\Phi(1) - \lambda I\right) &= \det \begin{pmatrix} (a-\lambda)I & -\omega I \\ aI & (1-\omega-\lambda)\, I \end{pmatrix} \\ &= \det \begin{pmatrix} I & 0 \\ \dfrac{a}{a-\lambda} I & I \end{pmatrix} \det \begin{pmatrix} (a-\lambda)I & -\omega I \\ 0 & \left(1-\omega-\lambda+\dfrac{a\omega}{a-\lambda}\right) I \end{pmatrix} \\ &= (a-\lambda)^n \left[(1-\omega-\lambda) + \frac{a\omega}{a-\lambda}\right]^n = \left[\lambda^2 - (1-\omega+a)\lambda + a\right]^n. \end{aligned} \tag{3.2}$$

Therefore, $\Phi(1)$ has at most two distinct eigenvalues, each with algebraic multiplicity n, i.e.

$$\begin{aligned} \lambda^{(1)} = \cdots = \lambda^{(n)} = \lambda_1 &= \frac{1-\omega+a-\left[(1-\omega+a)^2-4a\right]^{1/2}}{2} \\ \lambda^{(n+1)} = \cdots = \lambda^{(2n)} = \lambda_2 &= \frac{1-\omega+a+\left[(1-\omega+a)^2-4a\right]^{1/2}}{2}. \end{aligned} \tag{3.3}$$

Observe that either $\lambda_1 = a$ or $\lambda_2 = a$ *if and only if* $a = 0$ or $\omega = 0$. However, $a \neq 0$ and $\omega \neq 0$ is a very common assumption in PSO literature [CK02, KE95]. Thus, without loss of generality $\lambda_1 \neq a$ and $\lambda_2 \neq a$, and in accordance with the literature we may consider the following.

Assumption 3.1 *We assume in (3.1)* $a \neq 0$ *and* $\omega > 0$. □

Moreover, $p_\Phi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ is the minimal polynomial of matrix $\Phi(1)$ and $\det[\Phi(1)] = a^n$, so that for $n = 1$ we obtain $\det[\Phi(1)] = \lambda_1\lambda_2 = a$. By considering the quantity $\Delta = (1-\omega+a)^2 - 4a$ in (3.3), the following cases are analyzed:

- $a < 0$ which yields $\Delta > 0$, thus λ_1, λ_2 are *real* and *distinct* (one is positive and the other is negative);
- $a > 0$ which yields condition $\Delta \geq 0$ as long as (see Assumption 3.1)

$$0 < \omega \leq (1-\sqrt{a})^2 \qquad \text{or} \qquad \omega \geq (1+\sqrt{a})^2, \tag{3.4}$$

  which generates the following subcases:

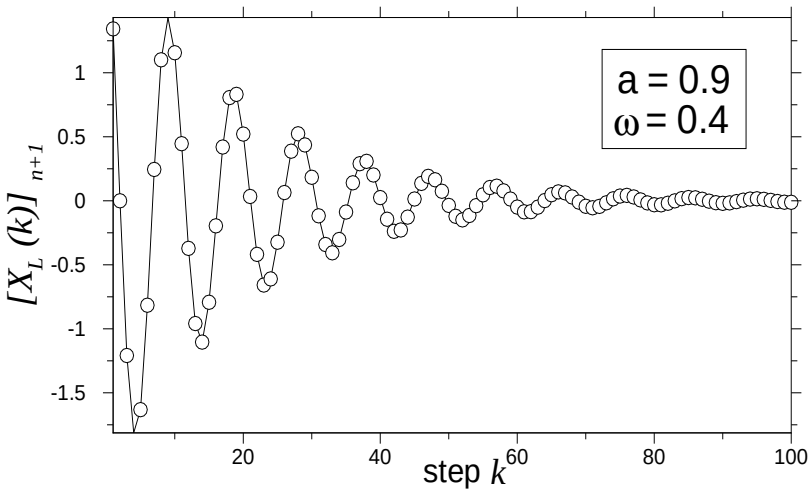

Figure 3.1: The $(n+1)$-th entry of $X_{\mathcal{L}}(k)$, with $0 < a < 1$ and $(1-\sqrt{a})^2 < \omega < (1+\sqrt{a})^2$. Eigenvalues λ_1 and λ_2 in (3.3) are complex conjugate: *pseudoperiodic modes*.

(1) $0 < \omega < (1-\sqrt{a})^2 \implies \lambda_1$ and λ_2 are real, distinct and both positive;

(2) $\omega = (1-\sqrt{a})^2$ or $\omega = (1+\sqrt{a})^2 \implies \lambda_1 = \lambda_2 = \dfrac{1-\omega+a}{2} = \pm\sqrt{a}$;

(3) $\omega > (1+\sqrt{a})^2 \implies \lambda_1$ and λ_2 are real, distinct and both negative.

To summarize the previous considerations in terms of dynamic analysis [S97] we have:

- if $a > 0$ and $(1-\sqrt{a})^2 < \omega < (1+\sqrt{a})^2$ then λ_1 and λ_2 are complex conjugate (see Fig. 3.1), so that they generate natural pseudoperiodic modes (this generalizes the results in [K98]). If $a > 0$ and $0 < \omega < (1-\sqrt{a})^2$ then we have exactly $2n$ natural aperiodic modes (see Fig. 3.2). Finally if $a > 0$ and $\omega > (1+\sqrt{a})^2$ then we have exactly $2n$ natural alternating modes (see Fig. 3.3);
- if $a < 0$ we have exactly n natural aperiodic modes and n natural alternating modes.

According with (2.6) [S97],

$$X_{\mathcal{L}}(k) = \Phi(k)X(0) = \Phi(1)^k X(0) = \begin{pmatrix} aI & -\omega I \\ aI & (1-\omega)\, I \end{pmatrix}^k X(0), \tag{3.5}$$

and from Assumption 3.1 $\lambda_q \neq a$, $q = 1, 2$, then

$$\operatorname{rk}\left[\Phi(1) - \lambda_q I\right] = \operatorname{rk}\left[\begin{pmatrix} I & 0 \\ \dfrac{a}{a-\lambda_q} I & I \end{pmatrix} \begin{pmatrix} (a-\lambda_q)I & -\omega I \\ 0 & 0 \end{pmatrix}\right] = n,$$

which implies that the *algebraic multiplicity* and the *geometric multiplicity* of eigenvalues λ_1 and λ_2 coincide with n, i.e. the unsymmetric matrix $\Phi(1)$ is *diagonalizable* and a basis of $2n$ real eigenvectors of $\Phi(1)$ exists.

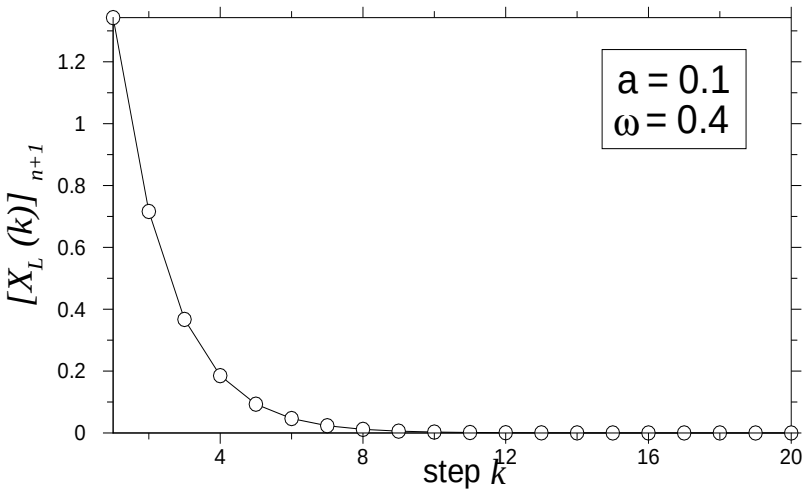

Figure 3.2: The $(n+1)$-th entry of $X_{\mathcal{L}}(k)$, with $0 < a < 1$ and $0 < \omega < (1-\sqrt{a})^2$. Eigenvalues λ_1 and λ_2 in (3.3) are real and positive: *aperiodic modes*.

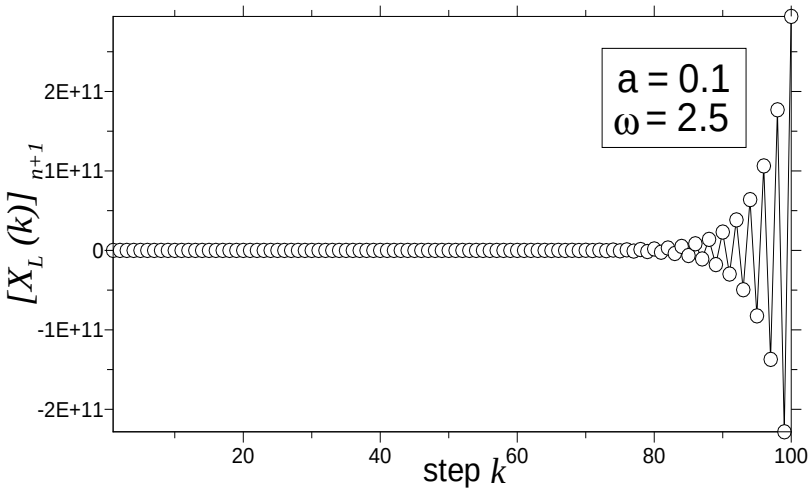

Figure 3.3: The $(n+1)$-th entry of $X_{\mathcal{L}}(k)$, with $0 < a < 1$ and $\omega > (1+\sqrt{a})^2$. Eigenvalues λ_1 and λ_2 in (3.3) are real and negative: *alternating modes*.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ (a-\lambda_1)/\omega \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \end{matrix} \quad , \quad \cdots \quad , \quad v_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ (a-\lambda_1)/\omega \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \end{matrix}$$

$$v_{n+1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ (a-\lambda_2)/\omega \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \end{matrix} \quad , \quad \cdots \quad , \quad v_{2n} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ (a-\lambda_2)/\omega \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n. \end{matrix}$$

Table 3.1: The $2n$ eigenvectors of matrix $\Phi(1)$ when Assumption 3.2 holds.

3.1 Computing the eigenvectors of $\Phi(1)$

According with the last considerations of Section 3, $2n$ eigenvectors of matrix $\Phi(1)$ exist and without loss of generality they have the form

$$v_i = \begin{pmatrix} v_i^1 \\ \\ v_i^2 \end{pmatrix}, \qquad v_i^1, v_i^2 \in \mathbb{R}^n, \quad i = 1, \ldots, 2n.$$

Then, to compute the eigenvectors of $\Phi(1)$ we impose relations

$$0 = \left[\Phi(1) - \lambda^{(i)} I\right] v_i = \begin{pmatrix} I & 0 \\ \\ \dfrac{a}{a-\lambda^{(i)}} I & I \end{pmatrix} \begin{pmatrix} (a-\lambda^{(i)})I & -\omega I \\ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i^1 \\ \\ v_i^2 \end{pmatrix},$$

so that from Assumption 3.1 we explicitly obtain the $2n$ eigenvectors in Table 3.1. These eigenvectors are linearly independent, provided that $\lambda_1 \neq \lambda_2$; the latter condition is not restrictive for our purposes, therefore from (3.4) we consider the following.

Assumption 3.2 *We assume in (3.1) $\omega \neq (1-\sqrt{a})^2$ and $\omega \neq (1+\sqrt{a})^2$ for any $a > 0$, so that the $2n$ eigenvectors of matrix $\Phi(1)$ are linearly independent.* □

Therefore, the matrix $V = [v_1 \cdots v_n v_{n+1} \cdots v_{2n}] \in \mathbb{R}^{2n \times 2n}$ exists, such that $V^{-1}\Phi(1)V = \Lambda$, with

$$\Lambda = \begin{pmatrix} \lambda_1 I & \\ & \lambda_2 I \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$$

and after few calculation

$$V = \begin{pmatrix} I & I \\ \\ \dfrac{a-\lambda_1}{\omega} I & \dfrac{a-\lambda_2}{\omega} I \end{pmatrix} \qquad \text{with} \qquad V^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} (a-\lambda_2)I & -\omega I \\ \\ -(a-\lambda_1)I & \omega I \end{pmatrix}.$$

$$\gamma_1(k) = \begin{cases} \dfrac{\lambda_1^k(a-\lambda_2) - \lambda_2^k(a-\lambda_1)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \ \text{ real} \\ \rho^{k+1}\dfrac{\sin k\theta}{\sin\theta} - \rho^k \dfrac{\sin(k-1)\theta}{\sin\theta} & \lambda_1, \lambda_2 \ \text{ complex conjugate,} \end{cases}$$

$$\gamma_2(k) = \begin{cases} \dfrac{\omega(\lambda_1^k - \lambda_2^k)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \ \text{ real} \\ \omega\rho^{k-1}\dfrac{\sin k\theta}{\sin\theta} & \lambda_1, \lambda_2 \ \text{ complex conjugate,} \end{cases}$$

$$\gamma_3(k) = \begin{cases} \dfrac{(\lambda_1^k - \lambda_2^k)}{\lambda_1 - \lambda_2}\dfrac{(a-\lambda_1)(a-\lambda_2)}{\omega} & \lambda_1, \lambda_2 \ \text{ real} \\ \rho^k \dfrac{\sin k\theta}{\sin\theta}\left(\dfrac{\rho^2 - 2\rho\cos\theta + 1}{\omega}\right) & \lambda_1, \lambda_2 \ \text{ complex conjugate,} \end{cases}$$

$$\gamma_4(k) = \begin{cases} \dfrac{\lambda_1^k(a-\lambda_1) - \lambda_2^k(a-\lambda_2)}{\lambda_1 - \lambda_2} & \lambda_1, \lambda_2 \ \text{ real} \\ \rho^{k+1}\dfrac{\sin k\theta}{\sin\theta} - \rho^k \dfrac{\sin(k+1)\theta}{\sin\theta} & \lambda_1, \lambda_2 \ \text{ complex conjugate.} \end{cases}$$

Table 3.2: The coefficients $\gamma_i(k)$, $i = 1, \ldots, 4$ in (3.6).

The previous results yield from (3.5)

$$\begin{aligned} X_{\mathcal{L}}(k) &= V\Lambda^k V^{-1} X(0) \\ &= \frac{1}{\lambda_1 - \lambda_2}\sum_{i=1}^{n}\left[\lambda_1^k X(0)^T\left((a-\lambda_2)e_i - \omega e_{n+i}\right)v_i - \lambda_2^k X(0)^T\left((a-\lambda_1)e_i - \omega e_{n+i}\right)v_{n+i}\right] \\ &= \frac{1}{\lambda_1 - \lambda_2}\sum_{i=1}^{n}\left[\lambda_1^k\left((a-\lambda_2)X(0)_i - \omega X(0)_{n+i}\right)v_i - \lambda_2^k\left((a-\lambda_1)X(0)_i - \omega X(0)_{n+i}\right)v_{n+i}\right], \end{aligned}$$

where $e_i \in \mathbb{R}^{2n}$ is the unit vector with 1 in position i, and $X(0)_i$ is the i-th entry of $X(0)$, i.e.

$$X(0) = \begin{pmatrix} X(0)_1 \\ \vdots \\ X(0)_{2n} \end{pmatrix}.$$

Furthermore, from Table 3.1 and for $i = 1, \ldots, n$

$$\begin{cases} v_i = e_i + \dfrac{a-\lambda_1}{\omega}e_{n+i} \\ \\ v_{n+i} = e_i + \dfrac{a-\lambda_2}{\omega}e_{n+i}, \end{cases}$$

then

$$X_{\mathcal{L}}(k) = \sum_{i=1}^{n}\left[\gamma_1(k)X(0)_i e_i - \gamma_2(k)X(0)_{n+i}e_i + \gamma_3(k)X(0)_i e_{n+i} - \gamma_4(k)X(0)_{n+i}e_{n+i}\right], \tag{3.6}$$

where coefficients $\gamma_i(k)$, $i = 1, \ldots, 4$ are given in Table 3.2. Observe that the first value of each coefficient $\gamma_1(k), \ldots, \gamma_4(k)$ refers to real eigenvalues λ_1 and λ_2, while the second value holds in case λ_1 and λ_2 are conjugate (i.e. $\lambda_1 = \rho e^{-j\theta}$, $\lambda_2 = \rho e^{j\theta}$, with $a = \rho^2$). In the end, from (3.6) we obtain

$$X_{\mathcal{L}}(k) = \sum_{i=1}^{n}\left[\gamma_1(k)X(0)_i - \gamma_2(k)X(0)_{n+i}\right]e_i + \left[\gamma_3(k)X(0)_i - \gamma_4(k)X(0)_{n+i}\right]e_{n+i}. \tag{3.7}$$

Remark 3.1 *Relation (3.7) suggests the following remarkable conclusions:*

- *the free response $X_{\mathcal{L}}(k)$ in (2.5) has the same formal expression for each particle, and is uniquely dependent on the initial point $X(0)$ of the particle;*
- *by simply imposing $X(0)_i = X(0)_{n+i} = 0$ in (3.7), the free response of a particle has zero entries on the i-th and $(n+i)$-th axis. Thus, according with the free response in (3.7), each particle's trajectory has nonzero projection on any subspace of $\mathbb{R}^{2n}$, provided that the initial point $X(0)$ is suitably chosen.*
- *if the eigenvalues λ_1 and λ_2 are complex conjugate, i.e. $\lambda_1 = \rho e^{-j\theta}$ and $\lambda_2 = \rho e^{j\theta}$, then the phase θ is given by*

$$\theta = \text{arctg}\left[\left(\frac{4a}{(1-\omega+a)^2} - 1\right)^{1/2}\right]. \tag{3.8}$$

Thus, the circular trajectories have the frequency $\theta/(2\pi)$, which might be suitably set in order to widely explore the state space. □

The importance of the latter considerations relies on the role played by $X_{\mathcal{L}}(k)$ and $X_{\mathcal{F}}(k)$ in the overall trajectory. From the expression of the coefficients $\gamma_1(k), \ldots, \gamma_4(k)$ in relation (3.7), it is clear that if $|\lambda_1| \geq 1$ or $|\lambda_2| \geq 1$ we might have $\|X_{\mathcal{L}}(k)\|_2 \to \infty$ for any infinite subsequence of k. Thus, in order to impose non-diverging trajectories, we have the following

Lemma 3.1 *Consider the eigenvalues λ_1 and λ_2 in (3.3) and let*

$$\begin{cases} |\lambda_1| < 1 \\ \\ |\lambda_2| < 1, \end{cases} \tag{3.9}$$

then

$$\lim_{k\to\infty} X_{\mathcal{L}}(k) = 0, \tag{3.10}$$

regardless of the choice of the initial point $X(0)$.

Proof
From (3.7) and Table 3.2, taking the limit $k \to \infty$ we obtain (3.10). □

From the latter lemma, observe that the free response $X_{\mathcal{L}}(k)$ of each particle does not affect the limit points of the sequence $\{X(k)\}$. However, a suitable choice of the starting point $X(0)$ of any particle may guarantee an improved exploration of the state space. The latter issue is noteworthy and plays a key role within global optimization frameworks, in order to ensure the convergence to any global minimum [P96].

In the light of the latter lemma the next proposition provides, under mild assumptions, a relation between the index k and the quantities $\|X_{\mathcal{L}}(k)\|_1$, $\|X_{\mathcal{L}}(0)\|_1$. We prove the latter result for a wide set of parameters a and ω in (3.1), which is relevant in the applications.

Proposition 3.2 *Let $0 < a < 1$ and $(1-\sqrt{a})^2 < \omega < (1+\sqrt{a})^2$ in (3.1). Let $0 < \epsilon_k < 1$ with $k \geq 0$, and assume*

$$\begin{aligned} k &\geq 1 + \frac{\ln\left(\frac{\omega|\sin\theta|\epsilon_k}{8}\right)}{\ln\rho} \qquad \text{for any } \ 0 < \omega < 1, \\ \\ k &\geq 1 + \frac{\ln\left(\frac{|\sin\theta|\epsilon_k}{8}\right)}{\ln\rho} \qquad \text{for any } \ \omega \geq 1, \end{aligned} \tag{3.11}$$

where $\lambda_1 = \rho e^{-j\theta}$ and $\lambda_2 = \rho e^{j\theta}$ are the eigenvalues of $\Phi(1)^k$. Then,

$$\|X_{\mathcal{L}}(k)\|_1 \leq \epsilon_k \|X_{\mathcal{L}}(0)\|_1. \tag{3.12}$$

Proof
The hypothesis straightforwardly yields $\rho = \sqrt{a} < 1$ and $0 < \omega < 4$. Then, from (3.6)

$$\|X_{\mathcal{L}}(k)\|_1 \;\leq\; \sum_{i=1}^{n} \left\{ \frac{\rho^{k-1}}{|\sin\theta|} \left[2|X_{\mathcal{L}}(0)_i| + 4|X_{\mathcal{L}}(0)_{n+i}|\right] + \frac{\rho^k}{|\sin\theta|} \left[\frac{4}{\omega}|X_{\mathcal{L}}(0)_i| + 2|X_{\mathcal{L}}(0)_{n+i}|\right] \right\},$$

which yields

$$\|X_{\mathcal{L}}(k)\|_1 \;\leq\; \frac{4\rho^{k-1}}{\omega|\sin\theta|} \left[2\|X_{\mathcal{L}}(0)\|_1\right] \;=\; \frac{8}{\omega}\frac{\rho^{k-1}}{|\sin\theta|}\|X_{\mathcal{L}}(0)\|_1, \qquad \text{for } 0 < \omega < 1,$$

$$\|X_{\mathcal{L}}(k)\|_1 \;\leq\; \frac{4\rho^{k-1}}{|\sin\theta|} \left[2\|X_{\mathcal{L}}(0)\|_1\right] \;=\; 8\frac{\rho^{k-1}}{|\sin\theta|}\|X_{\mathcal{L}}(0)\|_1, \qquad \text{for } \omega \geq 1.$$

Then, after few calculation we see that the inequalities

$$\frac{8}{\omega}\frac{\rho^{k-1}}{|\sin\theta|}\|X_{\mathcal{L}}(0)\|_1 \;\leq\; \epsilon_k\|X_{\mathcal{L}}(0)\|_1, \qquad \text{for } 0 < \omega < 1,$$

$$8\frac{\rho^{k-1}}{|\sin\theta|}\|X_{\mathcal{L}}(0)\|_1 \;\leq\; \epsilon_k\|X_{\mathcal{L}}(0)\|_1, \qquad \text{for } \omega \geq 1,$$

can be respectively satisfied if relations (3.11) hold. □

Finally observe that the equivalence among norms, provides a relation between the quantities $\|X_{\mathcal{L}}(k)\|_2$ and $\|X_{\mathcal{L}}(0)\|_2$, similar to (3.12).

4 Hints for the starting point of each particle

According with Remark 3.1, in this section we give some theoretical indications about the choice of the initial point $X(0)$ of each particle, which is a crucial issue in PSO. In particular, let the set $\mathcal{L}$ in (1.1) satisfy $\mathcal{L} = \{x \in \mathbb{R}^n : \|x\|_\alpha \leq L, \quad L > 0, \quad \alpha > 0\}$ and let us indicate $X(k)^{(j)}$ the trajectory of the j-th particle. From (2.5) $X(k)^{(j)}$ linearly depends on the contribution of the free response $X_{\mathcal{L}}(k)^{(j)}$, i.e. it depends on the initial point $X(0)^{(j)}$.

In this section we assume that exactly n particles compose the swarm, though with a similar reasoning our results may be extended to smaller/larger sets of particles. We aim at assessing the initial points $X(0)^{(j)}$, $j = 1, \ldots, n$, of the particles, in such a way that the state space $\mathbb{R}^{2n}$ is explored as widely as possible by the trajectories $\{X(k)^{(j)}\}$. We propose a set of starting points $\{X(0)^{(j)}\}$ such that for any fixed index k, the sequence $\{X_{\mathcal{L}}(k)^{(j)}\}$ is *scattered* in the state space $\mathbb{R}^{2n}$.
To the latter purpose we consider in (3.1) for any particle $j = 1, \ldots, n$ the parameters a_j, w_j in place of a, w. Accordingly, coefficients $\gamma_1(k), \ldots, \gamma_4(k)$ in (3.7) are given for the j-th particle by $\gamma_1(k)^{(j)}, \ldots, \gamma_4(k)^{(j)}$.

Consider the scalars $\alpha^{(j)}, \beta^{(j)} \in \mathbb{R}$, with $|\alpha^{(j)}| + |\beta^{(j)}| \neq 0$, and the vector $t_j \in \mathbb{R}^n$, $j = 1, \ldots, n$, such that

$$t_j = \frac{\sqrt{n}}{n}\sum_{i=1}^{n} e_i - \frac{\sqrt{n}}{2}e_j. \tag{4.1}$$

Then, let the vector $X(0)^{(j)}$ be given by

$$X(0)^{(j)} = \begin{pmatrix} \alpha^{(j)}t_j \\ \\ \beta^{(j)}t_j \end{pmatrix}, \qquad j = 1, \ldots, n. \tag{4.2}$$

Proposition 4.1 *Consider the set of initial points $\{X(0)^{(j)}\}$ defined in (4.2). Consider in (3.1) for any particle j the parameters a_j, w_j in place of parameters a, w, and let the Assumption 3.2 hold. Then for any $k \geq 0$ the following relations hold*

$$\left[X_{\mathcal{L}}(k)^{(j)}\right]^T X_{\mathcal{L}}(k)^{(h)} = 0, \qquad \text{for any } \; 1 \leq j \neq h \leq n. \tag{4.3}$$

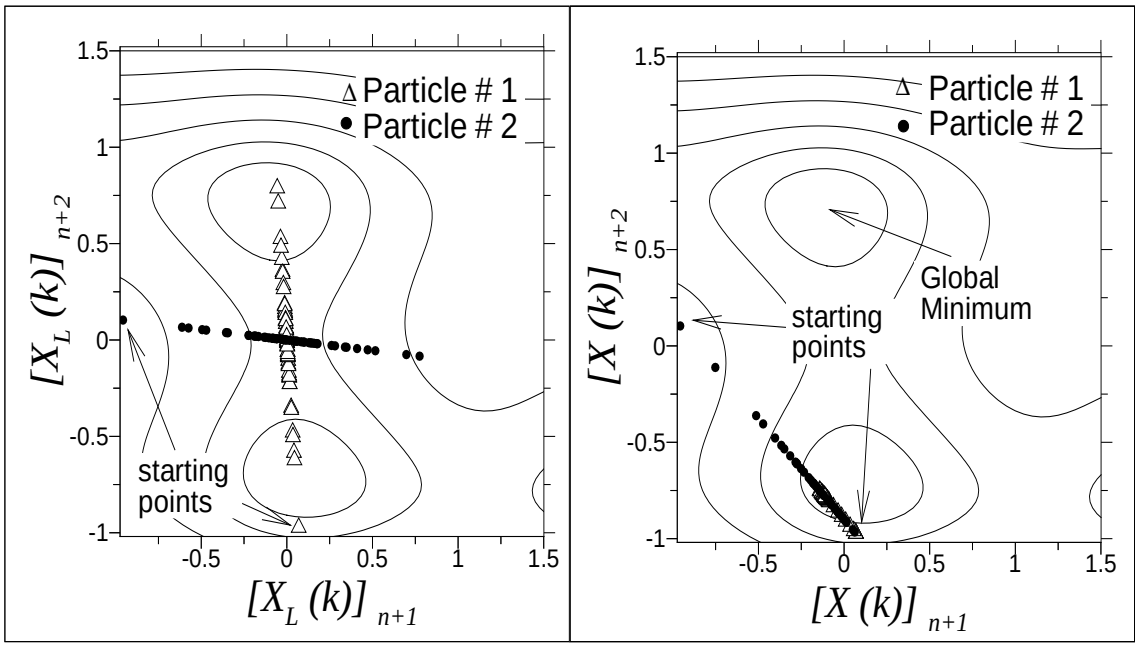

Figure 4.1: Random choice of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$ (*'Six-humps camel-back'* function [DS78]). Particle # 1 and Particle # 2 substantially overlap: the global minimum is not detected.

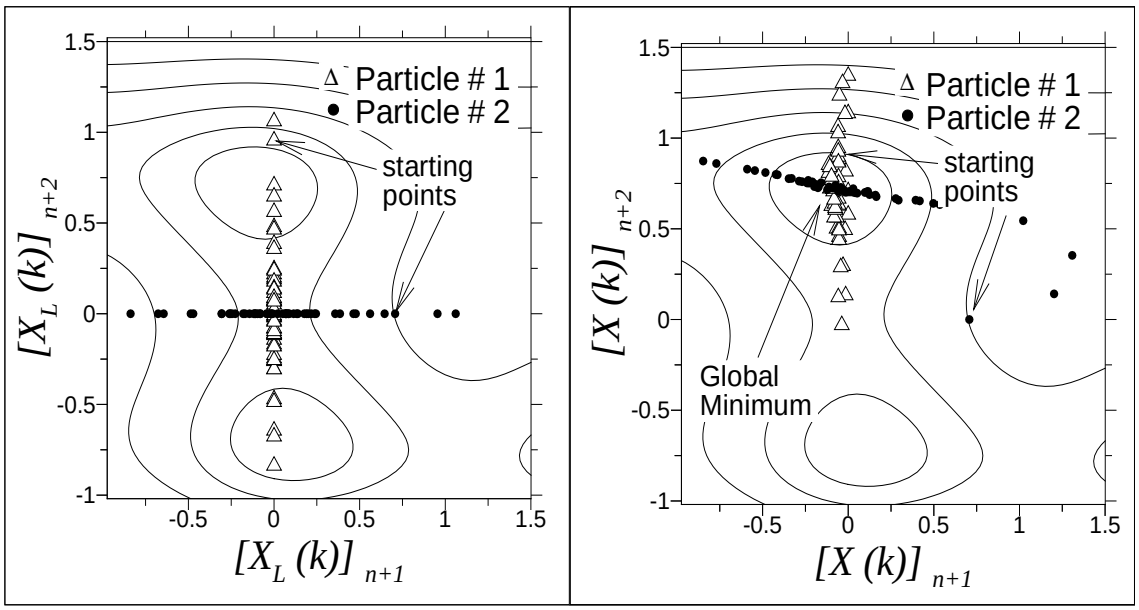

Figure 4.2: Choice (4.2) of initial points $X(0)^j$, $j = 1, 2$, with $n = 2$ (*'Six-humps camel-back'* function [DS78]). The two particles generate independent trajectories: the global minimum is detected.

Proof
From Assumption 3.2 relation (3.7) is defined. Thus, by simple substitution from (4.1) and (4.2), we obtain

$$X_{\mathcal{L}}(k)^{(j)} = \begin{pmatrix} \left[\alpha^{(j)}\gamma_1(k)^{(j)} - \beta^{(j)}\gamma_2(k)^{(j)}\right] t_j \\ \\ \left[\alpha^{(j)}\gamma_3(k)^{(j)} - \beta^{(j)}\gamma_4(k)^{(j)}\right] t_j \end{pmatrix}$$

$$X_{\mathcal{L}}(k)^{(h)} = \begin{pmatrix} \left[\alpha^{(h)}\gamma_1(k)^{(h)} - \beta^{(h)}\gamma_2(k)^{(h)}\right] t_h \\ \\ \left[\alpha^{(h)}\gamma_3(k)^{(h)} - \beta^{(h)}\gamma_4(k)^{(h)}\right] t_h \end{pmatrix},$$

and since $t_j^T t_h = 0$, for $1 \leq j \neq h \leq n$, relations (4.3) hold. □

We remark that the choice (4.1)-(4.2) generates at step k vectors $X_{\mathcal{L}}(k)^{(1)}, \ldots, X_{\mathcal{L}}(k)^{(n)}$, which form an orthogonal basis in the state subspace of the positions. In Figg. 4.1 and 4.2 we show an example with $n = 2$ (i.e. $X_{\mathcal{L}}(k)^{(j)} \in \mathbb{R}^4$, $j = 1, 2$, $k \geq 0$) and two particles (Particles # 1 and # 2 are sketched in the two dimensional subspace of positions with empty triangles and filled circles respectively; we set $a_1 = a_2 = 0.9$ and $\omega_1 = \omega_2 = 0.4$ in (3.1) for both particles). On the left sides of the figures only the free responses $X_{\mathcal{L}}(k)^{(1)}$ and $X_{\mathcal{L}}(k)^{(2)}$ are drawn, while on the right sides the full trajectories $X(k)^{(1)}$ and $X(k)^{(2)}$ are reported. Moreover, in Fig. 4.1 the initial points $X(0)^{(1)}$ and $X(0)^{(2)}$ are randomly chosen, while in Fig. 4.2 $X(0)^{(1)}$ and $X(0)^{(2)}$ are chosen as suggested in (4.2) (this explains the perfect orthogonality of the free responses). Observe that in the first case the algorithm fails to converge, because the trajectories substantially move along the segment joining $X(0)^{(1)}$ and $X(0)^{(2)}$. On the contrary, in Fig. 4.2 the orthogonality of the free responses of the two particles (left side) determines the convergence (Particle #2 is attracted by Particle #1 but from (4.3) they do not overlap). The latter result may be substantially interpreted as follows. At least for small values of parameter k, the choice (4.2) tends to preserve orthogonality among the trajectories. This helps the particles to be 'distributed' in the state space, inasmuch as for any k the set of positions $\{x_1^k, \ldots, x_P^k\}$ (see (2.3)) is likely an independent set.
Finally, we can readily prove that replacing the sequence $\{t_j\}$ in 4.1 with any orthogonal sequence of n nonzero vectors, the result of Proposition 4.1 still holds.

5 Conditions for non-diverging trajectories

In this section we consider the iteration (2.3), where the parameters χ_j^k, w_j^k, $c_{h,j}^k$, $r_{h,j}^k$, are possibly *non-constant* at each iteration[1] $k \geq 0$. Let the parameter w_j^k in (2.3) be given by

$$w_j^k = \phi_j(k) w_j^a + [1 - \phi_j(k)]\, w_j^b, \tag{5.1}$$

where $0 < \phi_j(k) \leq 1$, $w_j^b \geq 0$, $w_j^a \geq 0$. The expression in (5.1) includes the proposals for coefficient w_j^k in [SE98, KE95, CK02, ZMZQ03], where there is not full agreement about both the role and the way to assess w_j^k.
Regardless of possible heuristic interpretations of the role played by parameter w_j^k in (2.3), here we analyze the computation of the eigenvalues λ_1 and λ_2 in (3.3), when the coefficient w_j^k is possibly calculated as in (5.1). Therefore the coefficient a in (3.1) (as well as the coefficient ω and the eigenvalues λ_1 and λ_2) are affected by w_j^k, so that at step k we have to consider in (3.1) the parameters

$$\begin{aligned} a_j^k &= \chi_j^k w_j^k \\ \omega_j^k &= \sum_{h=1}^{p} \chi_j^k c_{h,j}^k r_{h,j}^k \end{aligned} \tag{5.2}$$

[1] This implies that the linear system (2.3) may not be time-invariant any more for the j-th particle. Anyway, from the *causality* property of system (2.6) [S97], for any particle we can equivalently consider at step k a new linear system, with the same parameters of (2.6), and whose starting point is $X(k-1)$ instead of $X(0)$. For the particle j the vector $X(k-1)$ is computed at step $k-1$ of (2.6), with the parameters χ_j^{k-1}, w_j^{k-1}, $c_{h,j}^{k-1}$, $r_{h,j}^{k-1}$.

in place of a and ω. Moreover, from (3.2) $\lambda_1\lambda_2 = a_j^k$ and from Lemma 3.1 we have to impose at each iteration conditions (3.9), in order to obtain (3.10). Therefore the parameter a_j^k should satisfy

$$|a_j^k| < 1. \tag{5.3}$$

This result is used in the following proposition.

Proposition 5.1 *Consider iteration (2.3) and relation (5.2), let at step $k \geq 0$*

$$\begin{aligned} &0 < |a_j^k| < 1, && j = 1, \ldots, P \\ &0 < \omega_j^k < 2(a_j^k + 1), && j = 1, \ldots, P, \end{aligned} \tag{5.4}$$

with $\omega_j^k \neq (1 \pm \sqrt{a_j^k})^2$ for any $a_j^k > 0$. Then, for any $k \geq 0$ and $1 \leq j \leq P$ we have $|\lambda_1| < 1$ and $|\lambda_2| < 1$ in relation (3.9).

Proof
We distinguish two cases: $-1 < a_j^k < 0$ and $0 < a_j^k < 1$. In the first case, imposing $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and considering that λ_1 and λ_2 are real, after some tedious calculation we obtain for $k \geq 0$ and $1 \leq j \leq P$ conditions

$$-1 < a_j^k < 0, \qquad 0 < \omega_j^k < 2(a_j^k + 1).$$

In the second case we consider two sub-cases:

(1) $0 < a_j^k < 1, \quad 0 < \omega_j^k < (1 - \sqrt{a_j^k})^2$ or $(1 + \sqrt{a_j^k})^2 < \omega_j^k < 2(a_j^k + 1)$,

(2) $0 < a_j^k < 1, \quad (1 - \sqrt{a_j^k})^2 < \omega_j^k < (1 + \sqrt{a_j^k})^2$.

If (1) holds λ_1 and λ_2 are real, so that again a long but trivial computation yields $|\lambda_1| < 1$ and $|\lambda_2| < 1$. On the other hand, if (2) holds then from (3.3) λ_1 and λ_2 are complex conjugate with $|\lambda_1| = |\lambda_2| = \sqrt{a_j^k} < 1$. This completes the proof. □

Remark 5.1 *From Proposition 5.1 observe that (5.3) is not a sufficient condition to impose non-diverging particle trajectories. Moreover, the particle trajectory might be non-diverging* as long as *an index $\bar{k}$ exists, such that the hypotheses in Proposition 5.1 hold for $k \geq \bar{k}$. Clearly for $\bar{k} = 0$ we again obtain the results of Proposition 5.1. On this guideline the results of this section extend those in* [SE98, CK02].

6 Computation of the forced response

In this section we focus on the forced response of the general particle, whose expression is given from (2.6) and (2.5) by

$$X_{\mathcal{F}}(k) = \sum_{\tau=0}^{k-1} H(k-\tau)U(\tau) = \sum_{\tau=0}^{k-1} \Phi(1)^{k-\tau-1}U(\tau), \tag{6.1}$$

so that, for $k \to \infty$ the sum in (6.1) represents a convolution between the impulsive response matrix $H(k) = \Phi(1)^{k-1} \in \mathbb{R}^{2n \times 2n}$ and the input vector $U(k) \in \mathbb{R}^{2n}$. The usual approach in literature for handling relation (6.1) is the introduction of a proper linear operator: the Z-transform [C99]. The Z-transform of a sequence of real matrices $\{M(k)\}$, $M(k) \in \mathbb{R}^{m \times q}$, $k \in \mathbf{Z}$, $m \geq 1$, $q \geq 1$, is the linear operator $F(z)$ such that $F : \{M(k)\} \to \mathbf{C}^{m \times q}$, with

$$F(z) = Z[M(k)] \stackrel{\text{def}}{=} \sum_{k=-\infty}^{+\infty} M(k) z^{-k}, \qquad z \in \mathbf{C}. \tag{6.2}$$

It can be proved [AM97] that for $k \to \infty$, by applying the Z-transform F to both the sides of (6.1), we can reduce the convolution to the simple matrix-vector product

$$\hat{X}_{\mathcal{F}}(z) = \hat{G}(z)\hat{U}(z), \tag{6.3}$$

where

$$\begin{aligned} \hat{X}_{\mathcal{F}}(z) &= Z\left[X_{\mathcal{F}}(k)\right] \\ \hat{G}(z) &= Z\left[\Phi(1)^{k-1}\right] \\ \hat{U}(z) &= Z\left[U(k)\right]. \end{aligned} \tag{6.4}$$

Observe that the power series (6.2) converges in a suitable *Region Of Convergence* (ROC) which depends on the sequence $\{M(k)\}$. In particular, the second relation (6.4) holds (i.e. the Z-transform $Z\left[\Phi(1)^{k-1}\right]$ converges) as long as

$$|z| > \|\Phi(1)\|_2. \tag{6.5}$$

In addition, it can be proved [S97] that introducing the Heavyside function

$$\delta_{-1}(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0, \end{cases}$$

we have

$$\hat{G}(z) = Z\left[\Phi(1)^{k-1}\right] = Z\left[\Phi(1)^{k-1}\delta_{-1}(k-1)\right] = \frac{Z\left[\Phi(1)^{k}\right]}{z} = (zI - \Phi(1))^{-1}. \tag{6.6}$$

Finally, by applying the inverse Z-transform [S97] to $\hat{X}_{\mathcal{F}}(z)$ we can compute $X_{\mathcal{F}}(k) = Z^{-1}\left[\hat{X}_{\mathcal{F}}(z)\right]$.

Furthermore, observe that the expression of $U(\tau)$ in relation (6.1) is not known, indeed from (2.9) $U(k)$ depends on $\{p_h^k\}$, $h = 1, \ldots, P$, thus its Z-transform $\hat{U}(z)$ may be of very difficult computation, even for special cases.
The difficult computation of $\hat{U}(z)$ implies that the convergence properties of the series in (6.1) can be hardly investigated. As a consequence, for any index $1 \leq j \leq P$ the convergence of the sequence $\{x_j^k\}$ cannot be ensured in the general case. Anyway, as recalled in the Introduction, PSO often provides a reasonable balance between the computational cost and the precision of the solution. Thus, it may be worth for several real problems (e.g. shape design optimization), which require a relatively small number of iterations. This implies that in (2.5) a range of values for k is considered, such that the free response $X_{\mathcal{L}}(k)$ is significantly bounded away from zero, and the leading role of the forced response $X_{\mathcal{F}}(k)$ should be reconsidered.
On this guideline, in Section 7 we will carry out some convergence results, which provide conditions for assessing the coefficients in iteration (2.3).

7 Preliminary convergence results

In this section we focus on problem (1.1) and we study some convergence properties of the sequences $\{x_j^k\}$, $j = 1, \ldots, P$ in (2.3), when no assumptions hold on $f(x)$, except the continuity. In respect to the previous section, here we also give conditions on the sequences $\{\chi_j^k\}$, $\{w_j^k\}$, $\{c_{h,j}^k\}$, $\{r_{h,j}^k\}$ in (2.3), which depend on indices $h, j = 1, \ldots, P$ and $k = 0, 1, \ldots$
We also highlight that in accordance with the final considerations of the previous section, here we substantially impose in our analysis some *conservative* conditions on the forced response $X_{\mathcal{F}}(k)$. In particular, in order to solve (1.1) with PSO, we have to guarantee that suitable PSO parameters exist such that the set $\mathcal{L}$ is bounded. This is achieved here by providing reasonable assumptions on PSO parameters, so that the robust and bounded set $\hat{\mathcal{L}}$ exists, which satisfies $\{x_j^k\} \subseteq \hat{\mathcal{L}}$, for any j and k.

Assumption 7.1 *Consider the global optimization problem (1.1) and iteration (2.3) with position (2.2). Let $\hat{k} \geq 0$ be an index such that for $j = 1, \ldots, P$, $\|x_j^{\hat{k}}\| \leq d$, $\|v_j^{\hat{k}}\| \leq d$ and $\|p_j^{\hat{k}}\| \leq d$, with $d < \infty$. Let $0 \leq \epsilon < 1$ and*

$$\hat{\mathcal{L}} = \left\{ y \in \mathbb{R}^n \ : \ \|y\| \leq \frac{d}{1-\epsilon} \right\}, \tag{7.1}$$

with $f(x)$ continuous *on the* robust *and* bounded set $\hat{\mathcal{L}}$. *Assume that for any* $k \geq \hat{k}$ *and* $j = 1, \ldots, P$ *the sequences* $\{\chi_j^k\}$, $\{w_j^k\}$, $\{c_{h,j}^k\}$, $\{r_{h,j}^k\}$ *satisfy:*

$$\text{(H1)} \qquad |\chi_j^k w_j^k| \leq \epsilon < 1,$$

$$\text{(H2)} \qquad \sum_{h=1}^{P} |\chi_j^k c_{h,j}^k r_{h,j}^k| \leq \frac{\left(\epsilon - |\chi_j^k w_j^k|\right) \epsilon^{k-\hat{k}}}{2/(1-\epsilon)}, \qquad h = 1, \ldots, P, \; p_h^k \neq x_j^k.$$

□

The next theorem proves that when solving (1.1), the PSO iteration (2.3) may generate feasible sequences of points $\{x_j^k\}$ in a suitable robust and bounded set $\hat{\mathcal{L}}$, such that $\hat{\mathcal{L}} \supseteq \mathcal{L}$.

Theorem 7.1 *Consider iteration (2.3) and suppose Assumption 7.1 holds. Then, for* $k \geq \hat{k}$ *we have:*

(a) $\{x_j^k\} \subset \hat{\mathcal{L}}, \;\; \{p_j^k\} \subset \hat{\mathcal{L}}, \qquad j = 1, \ldots, P,$

(b) $\|v_j^{k+1}\| \leq \epsilon^{k+1-\hat{k}} d, \qquad j = 1, \ldots, P,$

(c) $\lim_{k \to \infty} \|v_j^{k+1}\| = 0, \qquad j = 1, \ldots, P.$

Proof

Observe that by definition $p_h^k \in \mathcal{P}_h^k = \left\{ z \in \mathbb{R}^n \; : \; z = \arg\min_{\hat{k} \leq l \leq k} \{f(x_h^l), f(p_h^{\hat{k}})\} \right\}$, $h = 1, \ldots, P$ and by the definition of v_j^{k+1} in (2.3) we have

$$\|v_j^{k+1}\| \;\leq\; |\chi_j^k w_j^k| \, \|v_j^k\| + \left\| \sum_{h=1}^{P} \chi_j^k c_{h,j}^k r_{h,j}^k (p_h^k - x_j^k) \right\|, \qquad k \geq \hat{k}, \; j = 1, \ldots, P. \tag{7.2}$$

Now we prove *(a)* and *(b)* by induction. For $k = \hat{k}$ and $j = 1, \ldots, P$ we have

$$\begin{aligned} \|v_j^{\hat{k}+1}\| &\leq \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right| d + \sum_{h=1}^{P} \left|\chi_j^{\hat{k}} c_{h,j}^{\hat{k}} r_{h,j}^{\hat{k}}\right| \left\|p_h^{\hat{k}} - x_j^{\hat{k}}\right\| \;\leq\; \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right| d + \frac{\left(\epsilon - \left|\chi_j^{\hat{k}} w_j^{\hat{k}}\right|\right) 2d}{2/(1-\epsilon)} \;\leq\; \epsilon d, \\ \|x_j^{\hat{k}+1}\| &\leq \|x_j^{\hat{k}}\| + \|v_j^{\hat{k}+1}\| \;\leq\; (1+\epsilon) d, \end{aligned}$$

and $\|p_j^{\hat{k}+1}\| \leq \max\left\{\|p_j^{\hat{k}}\|, \|x_j^{\hat{k}+1}\|\right\} = (1+\epsilon)d$; thus, *(a)* and *(b)* hold with $k = \hat{k}$. Suppose now $x_j^k, p_j^k \in \hat{\mathcal{L}}$ and $\|v_j^k\| \leq \epsilon^{k-\hat{k}} d$, for $k > \hat{k}$; then, from Assumption 7.1 and relation (7.2):

$$\begin{aligned} \|v_j^{k+1}\| &\leq |\chi_j^k w_j^k| \, \|v_j^k\| + \frac{\left(\epsilon - |\chi_j^k w_j^k|\right) \epsilon^{k-\hat{k}} \left(\frac{2}{1-\epsilon}\right) d}{\left(\frac{2}{1-\epsilon}\right)} \\ &\leq |\chi_j^k w_j^k| \, \epsilon^{k-\hat{k}} d + \left(\epsilon - |\chi_j^k w_j^k|\right) \epsilon^{k-\hat{k}} d \;=\; \epsilon^{k+1-\hat{k}} d, \end{aligned} \tag{7.3}$$

$$\begin{aligned} \|x_j^{k+1}\| &= \left\| x_j^{\hat{k}} + \sum_{h=\hat{k}+1}^{k+1} v_j^h \right\| \;\leq\; \|x_j^{\hat{k}}\| + \sum_{h=\hat{k}+1}^{k+1} \|v_j^h\| \;\leq\; d + \sum_{h=\hat{k}+1}^{k+1} \epsilon^{h-\hat{k}} d \\ &\leq d + \frac{d}{\epsilon^{\hat{k}}} \left(\sum_{h=0}^{k+1} \epsilon^h - \sum_{h=0}^{\hat{k}} \epsilon^h \right) \;=\; d + \frac{d}{\epsilon^{\hat{k}}} \left(\frac{1-\epsilon^{k+2}}{1-\epsilon} - \frac{1-\epsilon^{\hat{k}+1}}{1-\epsilon} \right) \\ &= d + \frac{d}{\epsilon^{\hat{k}}} \left(\frac{\epsilon^{\hat{k}+1} - \epsilon^{k+2}}{1-\epsilon} \right) \;=\; d + \frac{\epsilon - \epsilon^{k+2-\hat{k}}}{1-\epsilon} d \;\leq\; d + \frac{\epsilon}{1-\epsilon} d \;=\; \frac{1}{1-\epsilon} d, \end{aligned} \tag{7.4}$$

so that again *(a)* and *(b)* hold. Finally, taking the limit in *(b)* we obtain *(c)*. Therefore, relations *(a)*, *(b)* and *(c)* hold for any infinite subsequence $\mathcal{K}_j \subseteq \{\hat{k}, \hat{k}+1, \ldots\}$ of index k. □

The hypotheses (H1)-(H2) at step $\hat{k}$ are theoretically effective in order to ensure that PSO iteration (2.3) is well suited for problem (1.1). However, (H1) is necessary from Proposition 5.1 and Lemma 3.1, but (H2) may be restrictive for several practical implementations, since it imposes for any $k \geq \hat{k}$ small stepsizes along the direction $(p_h^k - x_j^k)$ in (2.3). Anyway, since $\hat{k}$ is arbitrary, the latter drawback may be not so relevant.

Corollary 7.2 *Consider iteration (2.3) with position (2.2) and suppose Assumption 7.1 holds. Then, for* $j = 1, \ldots, P$*:*

1. *the sequence* $\{x_j^k\}$ *admits limit points and any limit point of* $\{x_j^k\}$ *belongs to* $\hat{\mathcal{L}}$*;*
2. *the sequences* $\{p_g^k\}$ *and* $\{p_j^k\}$ *admit limit points and any limit point of* $\{p_g^k\}$ *and* $\{p_j^k\}$ *belongs to* $\hat{\mathcal{L}}$*;*
3. *if* $\{x_j^k\}_{\mathcal{K}_j}$ *is an infinite subsequence of* $\{x_j^k\}$ *and* $\mathcal{K}_j \subseteq \{\hat{k}, \hat{k}+1, \ldots\}$*, then*

$$\lim_{k\to\infty,\ k\in\mathcal{K}_j} \|x_j^{k+1} - x_j^k\| = 0. \tag{7.5}$$

Proof
From *(a)* of Theorem 7.1 and the compactness of $\hat{\mathcal{L}}$ we obtain *1.*. As regards *2.*, it is an immediate consequence of *1.* and relations $\{p_g^k\} \subseteq \{p_j^k\} \subseteq \{x_j^k\} \subset \hat{\mathcal{L}}$. Finally, (2.3) and *(c)* of Theorem 7.1 give *3.*, for $j = 1, \ldots, P$. □

Observe that since relation (7.5) holds for *any* infinite subsequence $\mathcal{K}_j$ of $\{\hat{k}, \hat{k}+1, \ldots\}$, then also the following stronger relation holds:

$$\lim_{k\to\infty} \|x_j^{k+1} - x_j^k\| = 0, \qquad j = 1, \ldots, P.$$

The latter result substantially summarizes the convergence properties of the sequences $\{x_j^k\}$, $j = 1, \ldots, P$, in the compact set $\hat{\mathcal{L}}$. In other words, as long as the hypotheses of Theorem 7.1 hold, we can ensure that when solving (1.1) any subsequence $\{x_j^k\}_{\mathcal{K}_j} \subseteq \{x_j^k\}$ remains in $\hat{\mathcal{L}}$ and converges to a point of $\hat{\mathcal{L}}$. The following corollary completes the latter result. Moreover it provides suitable finite values of parameter ϵ in Theorem 7.1, in order to allow the convergence of any subsequence of $\{x_j^k\}$, $j = 1, \ldots, P$, to promising limit points.

Corollary 7.3 *Let* $\hat{k} \geq 0$ *be an index in (2.3) such that for* $j = 1, \ldots, P$, $\|x_j^{\hat{k}}\| \leq d$, $\|v_j^{\hat{k}}\| \leq d$ *and* $\|p_j^{\hat{k}}\| \leq d$*, with* $d < \infty$*. Let* $p_g^k \in \mathcal{P}_g^k = \left\{ z \in \mathbb{R}^n \ : \ z = \arg\min_{\hat{k}\leq l\leq k,\ h=1,\ldots,P} \{f(x_h^l), f(p_g^{\hat{k}})\} \right\}$*, for* $k \geq \hat{k}$*, and suppose* $\mathcal{K}_g \subseteq \{\hat{k}, \hat{k}+1, \ldots\}$ *is an infinite subsequence of indices such that*

$$\lim_{k\to\infty,\ k\in\mathcal{K}_g} x_j^k = \lim_{k\to\infty,\ k\in\mathcal{K}_g} p_g^k = p_g. \tag{7.6}$$

Then, if (H1)-(H2) *hold, for* $j = 1, \ldots, P$*, there exists a value* $\epsilon = \epsilon_g^j$ *such that*

$$\|x_j^{\hat{k}} - p_g\| = \lim_{k\to\infty,\ k\in\mathcal{K}_g} \left\| \sum_{i=\hat{k}}^{k} v_j^{i+1} \right\| = \lim_{k\to\infty,\ k\in\mathcal{K}_g} \sum_{i=\hat{k}}^{k} \left\| v_j^{i+1} \right\|. \tag{7.7}$$

Proof
From Theorem 7.1 we already know that $\{p_g^k\} \subset \hat{\mathcal{L}}$, where $\hat{\mathcal{L}}$ is defined in (7.1), so that the definition of $\mathcal{P}_g^k$ is well posed. Then, by the definition of x_j^{k+1} in (2.3) $\|x_j^{\hat{k}} - x_j^{k+1}\| = \left\| \sum_{i=\hat{k}}^{k} v_j^{i+1} \right\|$, moreover from (7.6) and Corollary 7.2

$$\lim_{k\to\infty,\ k\in\mathcal{K}_g} x_j^{k+1} \ = \ \lim_{k\to\infty,\ k\in\mathcal{K}_g} p_g^{k+1} \ = \ p_g \in \hat{\mathcal{L}},$$

thus

$$\lim_{k\to\infty,\ k\in\mathcal{K}_g} \|x_j^{\hat{k}} - x_j^{k+1}\| \ = \lim_{k\to\infty,\ k\in\mathcal{K}_g} \|x_j^{\hat{k}} - p_g^{k+1}\| \ = \ \|x_j^{\hat{k}} - p_g\| \ = \lim_{k\to\infty,\ k\in\mathcal{K}_g} \left\| \sum_{i=\hat{k}}^{k} v_j^{i+1} \right\|.$$

From *(b)* of Theorem 7.1

$$\begin{aligned}
\|x_j^{\hat{k}} - p_g\| &= \lim_{k\to\infty,\ k\in\mathcal{K}_g} \left\| \sum_{i=\hat{k}}^{k} v_j^{i+1} \right\| \leq \lim_{k\to\infty,\ k\in\mathcal{K}_g} \sum_{i=\hat{k}}^{k} (\epsilon_g^j)^{i+1-\hat{k}} d \\
&\leq (\epsilon_g^j)^{1-\hat{k}} d \lim_{k\to\infty,\ k\in\mathcal{K}_g} \left(\sum_{i=0}^{k} (\epsilon_g^j)^i - \sum_{i=0}^{\hat{k}-1} (\epsilon_g^j)^i \right) \\
&= (\epsilon_g^j)^{1-\hat{k}} d \left(\frac{1}{1-\epsilon_g^j} - \frac{1-(\epsilon_g^j)^{\hat{k}}}{1-\epsilon_g^j} \right) \ = \ \left(\frac{\epsilon_g^j}{1-\epsilon_g^j} \right) d. \qquad (7.8)
\end{aligned}$$

Thus, imposing in (7.8)

$$\epsilon_g^j = \frac{\|x_j^{\hat{k}} - p_g\|}{d + \|x_j^{\hat{k}} - p_g\|}, \qquad j = 1, \ldots, P$$

and observing that $\epsilon_g^j < 1$, we obtain (7.7). □

The following results complete our preliminary convergence analysis for iteration (2.3), and partially recall the analysis developed in Section 6. We highlight that in order to provide stronger convergence results with respect to Theorem 7.1 and its corollaries, in the following we reinforce the hypotheses.

Lemma 7.4 *Let λ_1 and λ_2 in (3.3) be the distinct eigenvalues of matrix $\Phi(1)$ in (2.7). If $|z| > \|\Phi(1)\|_2$, we have*

$$|z| > |\lambda_1| \quad \text{and} \quad |z| > |\lambda_2|.$$

Proof
For $\|\Phi(1)\|_2$ we have

$$\|\Phi(1)\|_2 \ \geq \ \inf_{\|\cdot\|_\beta} \{\|\Phi(1)\|_\beta\} \ = \ \max\{|\lambda_1|, |\lambda_2|\}.$$

Thus, the statement follows from the hypothesis. □

Consider relation (2.6); from the property of *stationarity* we obtain for $k > \hat{k} \geq 0$

$$\begin{aligned}
X(k) &= \Phi(k)X(0) + \sum_{\tau=0}^{k-1} H(k-\tau)U(\tau) \ = \ \Phi(k-\hat{k})X(\hat{k}) + \sum_{\tau=\hat{k}}^{k-1} H(k-\tau)U(\tau) \\
&= \Phi(k-\hat{k})X(\hat{k}) + \sum_{\bar{\tau}=0}^{k-\hat{k}-1} H(k-\hat{k}-\bar{\tau})U(\hat{k}+\bar{\tau}).
\end{aligned}$$

Moreover, in the hypotheses of Proposition 5.1, from Lemma 3.1, relation (2.5) yields for any infinite subsequence $\mathcal{K}$

$$\lim_{k\to\infty,\ k\in\mathcal{K}} X(k) = \lim_{k\to\infty,\ k\in\mathcal{K}} X_{\mathcal{F}}(k) = \lim_{k\to\infty,\ k\in\mathcal{K}} \sum_{\bar{\tau}=0}^{k-\hat{k}-1} H(k-\hat{k}-\bar{\tau})U(\hat{k}+\bar{\tau}).$$

Theorem 7.5 *Suppose Assumption 7.1 holds. Let in (2.3) and (2.2) for $j = 1, \ldots, P$*

$$\chi_j^k w_j^k = const., \quad \chi_j^k \neq 0, \qquad k > \hat{k},$$

$$\chi_j^k c_{h,j}^k r_{h,j}^k = 0, \quad \chi_j^k c_{g,j}^k r_{g,j}^k = const. \qquad k > \hat{k},\ h = 1, \ldots, P,\ h \neq g,\ p_h^k \neq x_h^k.$$

$$p_g^k = p_g \in \hat{\mathcal{L}}, \qquad k > \hat{k}.$$

Let Assumption 3.2 hold with $\omega = \sum_{h=1}^{P} \chi_j^k c_{h,j}^k r_{h,j}^k$ *and* $a = \chi_j^k w_j^k$, *for any* $k > \hat{k}$ *and* $j = 1, \ldots, P$. *Then,*

$$\lim_{k\to\infty} x_j^k = p_g, \qquad j = 1, \ldots, P. \tag{7.9}$$

In addition, if $\chi_j^k w_j^k = 0$, $k > \hat{k}$ *in (2.3), then the sequences* $\{x_j^k\}$, $j = 1, \ldots, P$, *are sublinearly convergent.*

Proof
From Section 6 the ROC of $\hat{G}(z)$ in (6.6) is given by $\{z \in \mathbf{C} \,:\, |z| > \|\Phi(1)\|_2,\ \gamma > 0\}$, so that from Lemma 7.4 we are ensured that $z \neq \lambda_1$ and $z \neq \lambda_2$. Thus, (6.3) and (6.6) yield for the j-th particle

$$(zI - \Phi(1))^{-1} = \begin{pmatrix} (z-a_j)I & \omega_j I \\ -a_j I & (z+\omega_j-1)I \end{pmatrix}^{-1} = \frac{1}{z^2 - (1-\omega_j+a_j)z + a_j} \begin{pmatrix} (z+\omega_j-1)I & -\omega_j I \\ a_j I & (z-a_j)I \end{pmatrix}, \tag{7.10}$$

where we set $a_j = \chi_j^k w_j^k$ and $\omega_j = \chi_j^k c_{g,j}^k r_{g,j}^k$, for any $k > \hat{k}$.
Furthermore, from the hypotheses and (2.9) we have in (6.3) [S97]

$$\hat{U}(z) = Z\left[\begin{pmatrix} \omega_j p_g \\ \omega_j p_g \end{pmatrix} \delta_{-1}(k)\right] = \frac{z}{z-1}\begin{pmatrix} \omega_j p_g \\ \omega_j p_g \end{pmatrix},$$

so that

$$\hat{X}_{\mathcal{F}}(z) = \frac{z}{z-1}(zI - \Phi(1))^{-1}\begin{pmatrix} \omega_j p_g \\ \omega_j p_g \end{pmatrix}. \tag{7.11}$$

Consequently, the matrices $R_0, R_1, R_2 \in \mathbf{C}^{2n\times 2n}$ exist such that from (3.2), (3.3), (7.10) and Assumption 3.2

$$\frac{z}{z-1}(zI - \Phi(1))^{-1} = \frac{R_0}{z-1} + \frac{R_1}{z-\lambda_1} + \frac{R_2}{z-\lambda_2},$$

where

$$\begin{aligned} R_0 &= \lim_{z\to 1}(z-1)\left[\frac{z}{(z-1)(z-\lambda_1)(z-\lambda_2)}\begin{pmatrix} (z+\omega_j-1)I & -\omega_j I \\ a_j I & (z-a_j)I \end{pmatrix}\right] \\ &= \frac{1}{(1-\lambda_1)(1-\lambda_2)}\begin{pmatrix} \omega_j I & -\omega_j I \\ a_j I & (1-a_j)I \end{pmatrix}, \end{aligned}$$

and similarly

$$\begin{aligned} R_1 &= \frac{\lambda_1}{(\lambda_1-1)(\lambda_1-\lambda_2)}\begin{pmatrix} (\lambda_1+\omega_j-1)I & -\omega_j I \\ a_j I & (\lambda_1-a_j)I \end{pmatrix}, \\ R_2 &= \frac{\lambda_2}{(\lambda_2-1)(\lambda_2-\lambda_1)}\begin{pmatrix} (\lambda_2+\omega_j-1)I & -\omega_j I \\ a_j I & (\lambda_2-a_j)I \end{pmatrix}. \end{aligned}$$

Finally, by applying the inverse Z-transform [S97] to $\hat{X}_{\mathcal{F}}(z)$ in order to get $X_{\mathcal{F}}(k) = Z^{-1}\left[\hat{X}_{\mathcal{F}}(z)\right]$, and recalling the property of stationarity, we simply obtain

$$X_{\mathcal{F}}(k) = Z^{-1}[\hat{X}_{\mathcal{F}}(z)] = \left[R_0 + (\lambda_1)^{k-\hat{k}}R_1 + (\lambda_2)^{k-\hat{k}}R_2\right]\begin{pmatrix} \omega_j p_g \\ \omega_j p_g \end{pmatrix}, \qquad k > \hat{k}.$$

Observe that from the hypotheses and Lemma 3.1 $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then

$$\lim_{k\to\infty} X(k) = \lim_{k\to\infty} X_{\mathcal{F}}(k) = R_0 \begin{pmatrix} \omega_j p_g \\ \\ \omega_j p_g \end{pmatrix}. \tag{7.12}$$

Thus, since $\omega_j/[(1-\lambda_1)(1-\lambda_2)] = 1$

$$R_0 \begin{pmatrix} \omega_j p_g \\ \\ \omega_j p_g \end{pmatrix} = \frac{\omega_j}{(1-\lambda_1)(1-\lambda_2)} \begin{pmatrix} \omega_j I & -\omega_j I \\ \\ a_j I & (1-a_j) I \end{pmatrix} \begin{pmatrix} p_g \\ \\ p_g \end{pmatrix} = \begin{pmatrix} 0 \\ \\ p_g \end{pmatrix},$$

which proves (7.9).

As regards the rate of convergence of the sequence $\{x_j^k\}$ in the index k, assuming also $\chi_j^k w_j^k = 0$ in (2.3) we obtain for $k > \hat{k}$

$$v_j^{k+1} = \omega_j (p_g - x_j^k)$$

$$x_j^{k+1} = x_j^k + v_j^{k+1},$$

which yields

$$\|x_j^{k+1} - p_g\| \ \leq \ \|x_j^k - p_g\| + |\omega_j| \|x_j^k - p_g\| \ = \ (1 + |\omega_j|) \|x_j^k - p_g\|,$$

and proves the *sublinear convergence* of the sequences $\{x_j^k\}$, $j = 1, \ldots, P$. □

We conclude the section by observing that from Theorem 7.5, the poor rate of convergence of the sequences $\{x_j^k\}$, $j = 1, \ldots, P$, suggests to combine the PSO iteration (2.3) with a faster local optimization method. Indeed, in order to speed up the convergence in the last iterations, the use of methods which efficiently exploit the local information of the objective function is advisable (see also [CLLPP05]).

8 Use of random coefficients in PSO algorithm

In (2.3) we considered a very general PSO iteration, which includes as special cases the proposals in [CK02, KE95]. Furthermore, in iteration (2.3) we provisionally assumed that the coefficients $r_{h,j}^k$, $h, j = 1, \ldots, P$ were *real constants* in $[0, 1]$, for any $k \geq 0$. In this section we generalize the latter assumption and consider that for $h, j = 1, \ldots, P$, the coefficient $r_{h,j}^k$ is a *random parameter* with *uniform distribution* in the interval $[0, 1]$. We aim at recasting the results of the previous sections with the latter new position[2].
On this guideline observe that in particular the analysis in Sections 3-5 still holds, since the inequalities involved are also satisfied with $r_{h,j}^k$ randomly distributed in $[0, 1]$. Indeed, whenever (5.4) holds with (see (5.2)) $r_{h,j}^k = 1$, then it also holds if $r_{h,j}^k$ is randomly distributed in $[0, 1]$, since a_j^k does not depend on $r_{h,j}^k$.

We only urge to precise, reasoning as in the proof of Proposition 5.1, that replacing '$r_{h,j}^k = 1$' with '$r_{h,j}^k$ randomly distributed in $[0, 1]$', may modify the natural modes associated with eigenvalues λ_1 and λ_2. Thus, Lemma 3.1 and Proposition 5.1 still hold: this result details more accurately the conclusions in [K98], i.e. *the above introduction of randomness for coefficients $r_{h,j}^k$, $h = 1, \ldots, P$, may not be responsible for instability on its own.* However, at step k random coefficients may cause complex eigenvalues λ_1 and λ_2 to become real, so that the particles trajectory may be modified during the iterations. This also implies that if the free response of the trajectory (2.3) of particle j is non-diverging with $r_{h,j}^k = 1$, then it also does not diverge with $r_{h,j}^k$ uniformly distributed in $[0, 1]$ (see for instance Figg. 8.1, ... where $r_{h,j}^k = 1$ -empty circles- and $r_{h,j}^k$ is randomly chosen in $[0, 1]$ -filled circles-). The latter conclusion is an immediate consequence of relation (5.4), where the sequences $\{\omega_j^k\}$ (and not directly $\{r_{h,j}^k\}$) are responsible to yield relation (3.9).

On the contrary, in those cases where the improper choice of deterministic coefficients in (2.3) causes diverging trajectories (empty circles in Fig. 8.2), the use of randomly chosen coefficient $r_{h,j}^k$ in $[0, 1]$ *may* yield convergence (filled circles in Figg. 8.2, ...).
Furthermore, the considerations on the forced response in Section 6 still apply with obvious modifications.

[2]The contents of footnote 1 apply here too, with trivial adaptations.

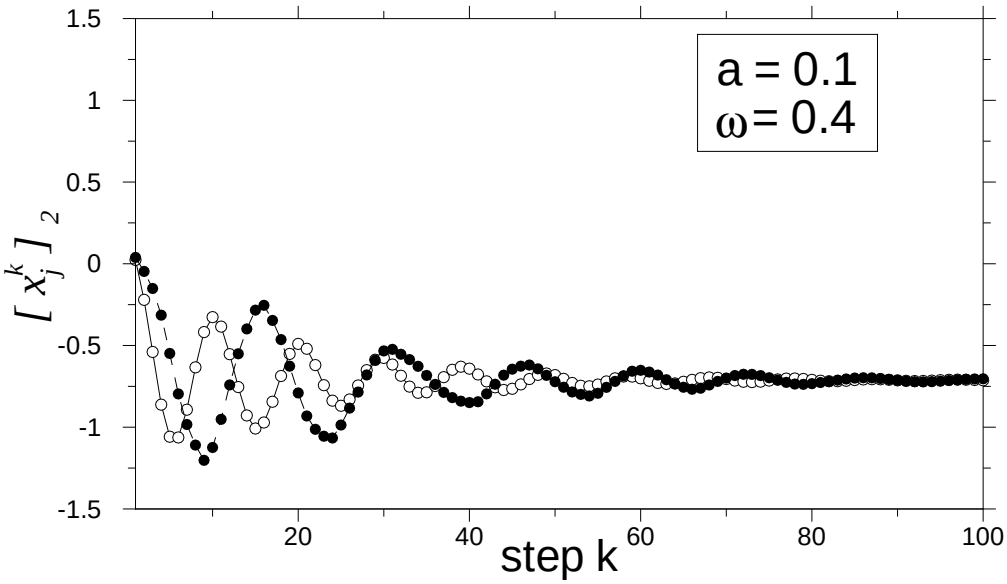

Figure 8.1: Comparison of the converging path of the component $\left[x_j^k\right]_2$, when choosing in (2.3) $r_{h,j}^k = 1$ (empty circles) and $r_{h,j}^k$ randomly distributed in $[0,1]$ (filled circles). Observe also that in the filled circles line, at each iteration the frequency is altered according with relation (3.8). Parameters a and ω in (3.1) are reported.

Of course these conclusions do not include any statistical analysis on the trajectory of the particles. In particular, we guess that a detailed investigation of the statistical properties of the sequences $\{v_j^k\}$ and $\{x_j^k\}$, $j = 1, \ldots, P$, in terms of the random sequences $\{r_{h,j}^k\}$, could provide helpful indications on determining the probability of convergence to global minima [P96]. In our opinion the latter issue deserves an appropriate study.

9 Conclusions and future work

In this paper a generalized version of Particle Swarm Optimization (PSO) algorithm is analyzed, and several conditions on its coefficients arise from our study. In particular, we provide sufficient conditions in order to avoid diverging paths of the particles, so that the PSO algorithm is suited for global optimization problem (1.1). Furthermore, our preliminary convergence analysis is in our knowledge one of the first attempts to investigate asymptotic properties of the particles trajectories, either in a deterministic or stochastic framework. This gives us a foundation for our future work, since a rigorous and (possibly) complete convergence analysis must be carried out. In particular, it is not clear the relation between the limit points of particles trajectories and the stationary points of the objective function to be minimized in (1.1).

Then, an extension to global optimization problems with special structure of the feasible set $\mathcal{L}$, seems the further necessary step. Indeed, in our opinion the PSO literature often contains examples of constrained problems where the use of heuristics covers up both aspects and limits of PSO iteration.

An extensive numerical experience, on both test problems and real applications of ship design, will be the key aspect of a forthcoming paper. There, we will use our theoretical results to give more precise indications on the choice of PSO coefficients. A comparison with the other proposals in the literature is also necessary, in order to test our approach in terms of efficiency and effectiveness.

We are also concerned with considering further generalizations of iteration (2.3), aiming at reducing the

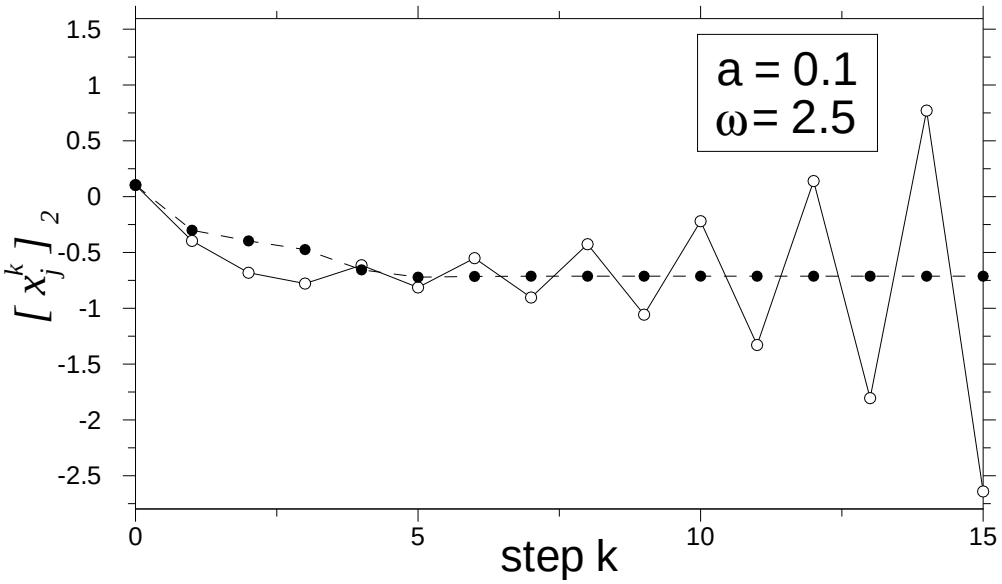

Figure 8.2: Path of the component $\left[x_j^k\right]_2$ when choosing in (2.3) $r_{h,j}^k = 1$ (*diverging* empty circles) and $r_{h,j}^k$ randomly distributed in $[0,1]$ (*converging* filled circles). Parameters a and ω in (3.1) are reported.

overall cost of the method. One possibility is given by updating the speed v_j^{k+1} in (2.3) as

$$v_j^{k+1} = \chi_j^k \left[w_j^k v_j^k + \sum_{h=1}^{P} c_{h,j}^k r_{h,j}^k (q_h^k - x_j^k) \right], \tag{9.1}$$

where

$$q_h^k = \text{argmin}_{l \in \mathcal{K}_h^k}\{f(x_h^l)\}, \qquad \mathcal{K}_h^k \subseteq \mathcal{K}^k = \{0,1,2,\ldots,k\}, \qquad h = 1,\ldots,P.$$

In other words we evaluate the objective function only at points in the subsets $\{q_h^k\}$, $h = 1,\ldots,P$, instead of considering the (possibly) larger sets $\{p_h^k\}$. We guess that this may imply a save in terms of function evaluations, whenever k increases. Thus, adopting an adaptive scheme to generate the subsets $\mathcal{K}_h^k$, $h = 1,\ldots,P$, the overall cost of the method might decrease. On the other hand the extension in (9.1) also allows more freedom of the particles trajectories, along with the possibility of skipping local minima. The latter approach suggests a further investigation in the context of *derivative free* methods, not to mention the more general context of *direct search* methods [KLT03] (see also [GP05]).

Finally, the possibility of clustering the particles, in order to possibly detect multiple global minima, is another issue of interest.

Acknowledgement

This work has been supported by the *Ministero delle Infrastrutture e dei Trasporti* in the framework of the research plan "Programma di Ricerca sulla Sicurezza", Decreto 17/04/2003 G.U. n. 123 del 29/05/2003.

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